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MATHEMATICS

On Some Geometrical Properties of Normal Matrices

by

S. GLIDMAN

Presented by A. MOSTOWSKI on June 16, 1961

1. Let

$$A = [\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n] \tag{1}$$

denote a normal real matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues ordered in such a way that

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|. \tag{2}$$

Let

$$G = A^T A = AA^T = [\bar{a}_i \cdot \bar{a}_k] \tag{3}$$

be the Gram matrix of A (by $\bar{a}_i \cdot \bar{a}_k$ we denote the scalar product of the vectors $\bar{a}_i$ and $\bar{a}_k$). We have

$$\bar{e}_i^T G \bar{e}_i = a_i^2, \tag{4}$$

where $\bar{e}_i$ is the i-th column of the unit matrix E. (We put $\bar{a}_i \cdot \bar{a}_i = a_i^2$). Hence, by the theorem of the extremum properties of eigenvalues [8], we obtain

$$|\lambda_n| \leqslant a_i \leqslant |\lambda_1|. \tag{5}$$

Inequality (5) can be easily generalized. Let $A^{(p)}$ be the p-th adjugate matrix of A [7], formed by the lexicographically arranged minors of order p of the matrix A.

It is easy to prove [7] that the p-th adjugate matrix $A^{(p)}$ of the normal matrix A is normal.

Let $\lambda_i^{(p)} \left(i = 1, 2, \ldots, \binom{n}{p}\right)$ be the eigenvalues of the matrix $A^{(p)}$. Then [6]

$$\lambda_i^{(p)} = \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_p} \tag{6}$$

where $(i_1, i_2, \ldots, i_p)$ is some combination of the elements $1, 2, \ldots, n$.

Let $\bar{a}_i^{(p)}$ be the successive i-th column of the matrix $A^{(p)}$. Let us assume that $\bar{a}_i^{(p)}$ contains the minors formed by the columns with indices $i_1, i_2, \ldots, i_p$ of matrix A. Using the outer product of Grassman [6] we have

$$\bar{a}_i^{(p)} = \bar{a}_{i_1} \wedge \bar{a}_{i_2} \wedge \ldots \wedge \bar{a}_{i_p}. \tag{7}$$

But, by the Cauchy theorem on the minors of a product matrix [8], we have

$$[a_i^{(p)}]^2 = G_{i,i_2 \dots i_p} \tag{8}$$

where $G_{i,i_2 \dots i_p}$ denotes the principal minor of order p of G formed by the columns with the indices $i_1, i_2, \dots, i_p$ of A. Thus, applying (5) to the matrix $A^{(p)}$ we get for $p = 1, 2, \dots, n$, $i = 1, 2, \dots, \binom{n}{p}$

$$|\lambda_{n-p+1}\, \lambda_{n-p+2} \dots \lambda_n| \leqslant \sqrt{G_{p,i}} \leqslant |\lambda_1\, \lambda_2 \dots \lambda_p|, \tag{9}$$

where $G_{p,i}$ is a principal minor of order p of G.

2. Let $S = [\bar{s}_1, \bar{s}_2, \dots, \bar{s}_n]$ be a symmetric matrix having eigenvalues $t_1, t_2, \dots, t_n$ ordered as follows

$$|t_1| = |t_2| = \dots = |t_{r_1}| > |t_{r_1+1}| = \dots = |t_{r_2}| > \dots > |t_{r_{m-1}+1}| = \dots = |t_n| \tag{10}$$

and let $\bar{x}_i = [x_{1i}, x_{2i}, \dots, x_{ni}]$ be the unit eigenvectors of S. The modal-matrix $X = [\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n]$ is then orthogonal. We have obviously $X\bar{e}_i = \bar{x}_i$ and $S\bar{x}_i = t_i \bar{x}_i$. Thus, we obtain

$$S^k \bar{e}_i = \bar{s}_{ki} = \sum_{j=1}^{n} t_j^k\, x_{ij}\, \bar{x}_j\,. \tag{11}$$

Hence, as X is orthogonal, we obtain

$$s_{ki}^2 = \sum_{j=1}^{n} t_j^{2k}\, x_{ij}^2 \tag{12}$$

or, by (10),

$$s_{ki}^2 = \sum_{p=1}^{m} c_{ip}\, t_{r_p}^{2k}\,, \tag{13}$$

where

$$c_{i,\,p+1} = \sum_{j=r_p+1}^{r_{p+1}} x_{ij}^2 \tag{14}$$

(we set for symmetry $r_0 = 0, r_m = n$).

It is easy to prove that if we order the columns of S^k according to their length this order remains invariant for sufficiently great values of k and depends only on the coefficients c_{ip}. Thus, there exists at least one column of S^k satisfying for sufficiently great k and for each $i = 1, 2, \dots, n$ the inequality $s_{k\alpha_i} \geqslant s_{ki}$.

The column of S with index α_1 will be called a definitively maximal column of S.

It has been proved that a definitively maximal column of a symmetric matrix is not orthogonal to at least one eigenvector corresponding to the dominant eigenvalue t_1.

If $c_{i1} \neq 0$, then, by (13) and (5), we obtain

$$\lim_{k \to \infty} \sqrt[k]{s_{ki}} = |t_1|\,. \tag{15}$$

Therefore

$$\lim_{k\to\infty} \sqrt[k]{s_{k\alpha_1}} = |t_1|. \tag{16}$$

Applying (16) to the Gram matrix of the p-th adjugate matrix $(A^k)^{(p)}$ it has been proved that

$$|\lambda_1 \lambda_2 \dots \lambda_p| = \lim_{k\to\infty} \sqrt[2k]{G^k_{p,\alpha_p}}\,, \tag{17}$$

where G^k_{p,α_p} denotes the definitively maximal principal minor of order p of the matrix G^k (for sufficiently great values of k the minor not smaller than the other minors is definitively maximal).

It has been proved that the sequences occurring in formulas (15) and (17) are non-decreasing.

3. Let us assume that for a non singular symmetric matrix S we have found such a permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ of the elements $1, 2, \dots, n$ that for $k > K$ and for each $p = 1, 2, \dots, n-1$ the condition

$$(\bar{s}_{k\sigma_1} \wedge \bar{s}_{k\sigma_2} \wedge \dots \wedge \bar{s}_{k\sigma_p}) \cdot (\bar{x}_1 \wedge \bar{x}_2 \wedge \dots \wedge \bar{x}_p) \neq 0 \tag{18}$$

holds (for $p = n$ this condition is always satisfied).

The permutation σ will be called the regular column-order of S.

Let us assume that the eigenvalues of S differ in absolute value and $|t_1| > |t_2| > \dots > |t_n| > 0$.

Considering the columns of $(S^k)^{(p)}$ $(p = 1, 2, \dots, n-1)$ formed by the columns of S^k occurring in the regular column-order σ we get from Mises' theorem [1]

$$\bar{s}_{k\sigma_1} \wedge \bar{s}_{k\sigma_2} \wedge \dots \wedge \bar{s}_{k\sigma_p} = (t_1 t_2 \dots t_p)^k (b_{kp}\, \bar{x}_1 \wedge \bar{x}_2 \wedge \dots \wedge \bar{x}_p + \bar{\beta}_{kp}), \tag{19}$$

where the length of the vector $\bar{\beta}_{kp}$ converges to zero while $k \to \infty$

From (19) and from the properties of the outer vector [6] it follows easily by induction that the column $\bar{s}_{k\sigma_i}$ $(i = 1, 2, \dots, n)$ takes then the form

$$\bar{s}_{k\sigma_i} = t_i^k (m_{ki1}\, \bar{x}_1 + m_{ki2}\, \bar{x}_2 + \dots + m_{kii}\, \bar{x}_i + \bar{\varepsilon}_{ki}), \tag{20}$$

where the length of the vector $\bar{\varepsilon}_{ki}$ converges to zero, if $k \to \infty$.

Let Q be the permutation matrix [1]

$$Q = [\bar{e}_{\sigma_1}, \bar{e}_{\sigma_2}, \dots, \bar{e}_{\sigma_n}]. \tag{21}$$

The successive columns of the matrix

$$P_k = S^k Q \tag{22}$$

are equal to those of S^k with the indices $\sigma_1, \sigma_2, \dots, \sigma_n$. Therefore, from (20) we have

$$P_k \cong X \Lambda_k D^k, \tag{23}$$

where

$$\Delta_k = \begin{bmatrix} m_{k11} & m_{k21} & \dots & m_{kn1} \\ & m_{k22} & \dots & m_{kn2} \\ & & \ddots & \vdots \\ & & & m_{knn} \end{bmatrix} \tag{24}$$

is an upper triangular matrix and D is a diagonal one

$$D = \operatorname{Diag}[t_1, t_2, \dots, t_n]. \tag{25}$$

The formula (23) gives the decomposition of P_k into the product of an orthogonal and an upper triangular matrix. Hence, it follows [5] that the successive altitudes of the matrix P_k are parallel to the successive columns of X. (By the p-th successive altitude of the matrix (1) we understand the vector of p-dimensional space $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_p)$, orthogonal to the hyperplane $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{p-1})$, of a length equal to the distance of the endpoint of $\bar{a}_p$ from this hyperplane and directed to this endpoint).

Thus, we have the following

THEOREM 1. *If S is a nonsingular symmetric matrix having eigenvalues of different absolute value ordered according to their decreasing absolute value, then the direction of the successive p-th altitude of S^k determined for a regular column-order converges for $k \to \infty$ to the direction of the p-th eigenvector of S.*

By a regular column-order $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ of a normal matrix A we understand the regular column-order of its Gram matrix.

From (7), (8), (15), (18) we obtain easily

$$|\lambda_1 \lambda_2 \dots \lambda_p| = \lim_{k\to\infty} \sqrt[2k]{G^k_{\gamma_1 \gamma_2 \dots \gamma_p}}, \tag{26}$$

where $G^k_{\gamma_1 \gamma_2 \dots \gamma_p}$ denotes the principal minor of order p of G^k formed by the columns of A^k with indices $\gamma_1, \gamma_2, \dots, \gamma_p$. Hence,

$$|\lambda_p| = \lim_{k\to\infty} \sqrt{\frac{G^k_{\gamma_1 \gamma_2 \dots \gamma_p}}{G^k_{\gamma_1 \gamma_2 \dots \gamma_{p-1}}}}. \tag{27}$$

Setting

$$h_{kp}(\gamma) = \sqrt{\frac{G^k_{\gamma_1 \gamma_2 \dots \gamma_p}}{G^k_{\gamma_1 \gamma_2 \dots \gamma_{p-1}}}} \tag{28}$$

we have

$$|\lambda_p| = \lim_{k\to\infty} \sqrt[k]{h_{kp}(\gamma)}. \tag{29}$$

It is known [3], that $h_{kp}(\gamma)$ is the length of the p-th successive altitude of the matrix formed by the columns $\bar{a}_{k\gamma_1}, \bar{a}_{k\gamma_2}, \dots, \bar{a}_{k\gamma_p}$ of A^k.

Thus, we have the following

THEOREM 2. *The absolute value of the p-th eigenvalue of a normal matrix A having its eigenvalues ordered according to their decreasing absolute value is the limit for* $k \to \infty$ *of the sequence* $\sqrt[k]{h_{kp}(\gamma)}$ *where* $h_{kp}(\gamma)$ *denotes the length of the successive p-th altitude of the matrix* A^k *formed for a regular column-order.*

If we apply formula (11) to the columns $\bar{s}_{k\sigma_1}, \bar{s}_{k\sigma_2}, \ldots, s_{k\sigma_p}$ of S^k, then, from the properties of the outer product [6] it follows that the regularity condition (18) is equivalent to the condition

$$|X_{p(\sigma)}| \neq 0 \tag{30}$$

for $p = 1, 2, \ldots, n-1$, where

$$X_p(\sigma) = \begin{bmatrix} x_{\sigma_1 1} x_{\sigma_2 2} \ldots x_{\sigma_1 p} \\ x_{\sigma_2 1} x_{\sigma_2 2} \ldots x_{\sigma_2 p} \\ \ldots\ldots\ldots \\ x_{\sigma_p 1} x_{\sigma_p 2} \ldots x_{\sigma_p p} \end{bmatrix}. \tag{31}$$

Hence, it has been proved by the well known Sylvester theorem [2] concerning the matrix formed of minors of a given matrix, that for a symmetric nonsingular matrix having eigenvalues of different absolute value one can effectively determine at least one regular column-order.

The normal matrix will be called regular, if the natural succession of its columns is a regular one.

We shall show that the matrix

$$S_1 = Q^T S Q, \tag{32}$$

where Q is the matrix (21), is regular. It is sufficient to note that for the modal-matrix $X(S_1)$ of S_1 we have [1]

$$X(S_1) = Q^T X. \tag{33}$$

Hence, by (31), we obtain

$$X(S_1) = X_n(\sigma), \tag{34}$$

therefore the matrix S_1 is regular.

For a symmetric regular matrix S we obtain from (22) and (23)

$$S^k \cong X \Delta_k D^k. \tag{35}$$

Thus, we have the following

THEOREM 3. *If we decompose the matrix* S^k, *where S denotes a symmetric nonsingular and regular matrix with eigenvalues of different absolute value, to the product of an orthogonal and an upper triangular matrix, then the successive columns of the orthogonal part in this decomposition, for* $k \to \infty$, *will be parallel to the successive eigenvectors of S.*

5. Let us express the nonsingular matrix A_1 as a product

$$A_1 = N_1 \delta_1, \tag{36}$$

where N_1 is an orthogonal, and δ_1 — an upper triangular matrix.

It is known [5] that the decomposition (36) will be unique if we determine the directions of the rows of δ_1 or of the columns of N_1. Choosing the second possibility we determine the directions of the columns of N_1 in such a way that their first significant components be positive.

Let us apply the LR Rutishauser algorithm to the now unique decomposition (36). We form the sequence of matrices

$$A_{k+1} = \delta_k N_k = N_{k+1} \delta_{k+1}. \tag{37}$$

Let us denote briefly

$$X_k = N_1 N_2 \ldots N_k, \tag{38}$$

$$\Delta_k = \delta_k \delta_{k-1} \ldots \delta_1. \tag{39}$$

Then we have

$$A_{k+1} = X_k^T A_1 X_k = \Delta_k A_1 \Delta_k^{-1} \tag{40}$$

and

$$A_1^k = X_k \Delta_k . \tag{41}$$

From the assumption that there exists $\lim X_k = X$ follows $\lim N_k = E$. Therefore, by (40), there exists $\lim A_k$, too.

Then from (37) we have

$$\lim A_k = \lim \delta_k = \delta, \tag{42}$$

where δ is, of course, an upper triangular matrix.

We shall prove that, if A_1 is a symmetric nonsingular and regular matrix with eigenvalues of different absolute value, then the sequence of the matrices A_k is convergent and the limit (42) exists.

From Theorem 3 and (41) it follows that for sufficiently great values of k the columns of X_k differ from those of X_{k-1} at most by their directions. Then, by (38), it follows that

$$N_k \approx P_k, \tag{43}$$

where P_k is a diagonal matrix the elements of which on the diagonal equal 1 or -1. Hence, for the chosen way of unique decomposition to the product of an orthogonal and a triangular matrix, we obtain

$$\lim N_k = E. \tag{44}$$

Thus, in this case the sequence (37) is convergent.

As follows from (40), the matrices A_k are then symmetric. Thus, from (42) it results that δ is a diagonal matrix having on its diagonal the eigenvalues of the given matrix.

THEOREM 4. *If we apply the LR algorithm to a definite unique decomposition of a symmetric nonsingular and regular matrix A with eigenvalues of different absolute value to the product of an orthogonal matrix N_1 and a triangular matrix δ_1, forming the sequence of matrices $A_{k+1} = \delta_k N_k = N_{k+1} \delta_{k+1}$, then there exists* $\lim A_k = \delta$, *where δ is a diagonal matrix similar to the given matrix.*

The paper will appear in full in Zeszyty Naukowe Politechniki Warszawskiej, Geodezja, No. 7.

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MATHEMATICS

Concerning the Method of Tangent Hyperbolas for Operator Equations

by

M. ALTMAN

Presented by W. ORLICZ on July 1, 1961

G. S. Salehov [5] investigated the method of tangent hyperbolas for real or complex function equations. He gave sufficient conditions for the convergence of the method to a solution which exists under the same conditions. An abstract variant of the method of tangent hyperbolas for operator equations is given by M. A. Mertvetzova [4] who extended the results of Salehov.

In this paper we investigate the operator variant of the method of tangent hyperbolas by using the majorant principle. An application of this principle to the functional variant of the method of tangent hyperbolas is given in [1]. This abstract variant is essentially different from the operator variant and concerns finding the zero elements of a non-linear functional in a Banach space.

Consider the operator equation

$$P(x) = 0, \tag{1}$$

where P is a non-linear operator from a Banach space X to a Banach space Y. Suppose that $P(x)$ possesses the derivatives of the first, second and third order in the sense of Fréchet, which are denoted by $P'(x)$, $P''(x)$ and $P'''(x)$, respectively. The continuity of the derivatives is required.

The method of tangent hyperbolas for the operator Eq. (1) is then defined as follows:

$$x_{n+1} = x_n - Q_n \Gamma_n P(x_n), \quad n = 0, 1, 2, \ldots, \tag{2}$$

where x_0 is the initial approximate solution of Eq. (1) and $\Gamma_n = [P'(x_n)]^{-1}$ denotes the inverse of $P'(x_n)$;

$$Q_n = \left[I - \frac{1}{2} \Gamma_n P''(x_n) \Gamma_n P(x_n) \right]^{-1}.$$

We shall now apply the majorant principle to investigate the method defined by formula (2).

For this purpose let us consider the real equation

$$f(z) = 0, \tag{3}$$

where $f(z)$ is a real function of the real variable z, being three times continuously differentiable in (z_0, z').

Following Kantorovitch [3] let us say that Eq. (1) possesses a real majorant equation (3), if the following conditions are satisfied

1° $\|P(x_0)\| \leqslant f(z_0)$;

2° $\|\Gamma_0\| \leqslant B_0$,

where

$$B_0 = -\frac{1}{f'(z_0)} > 0;$$

3° $\|P''(x)\| \leqslant f''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0$;

4° $\|P'''(x)\| \leqslant f'''(z)$ if $\|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0$;

Let us now construct the sequence of the approximate solutions z_n of Eq. (3) by the method of tangent hyperbolas:

$$z_{n+1} = z_n - \frac{1}{1 - \frac{1}{2}\frac{f''(z_n) f(z_n)}{[f'(z)]^2}} \frac{f(z_n)}{f'(z_n)}, \qquad n = 0, 1, 2, \ldots \tag{4}$$

The following simple lemma will be applied:

LEMMA. *Let us assume that the function $f(z)$ satisfies the following conditions*:

$$f(z) \geqslant 0 \quad \text{for} \quad z_0 \leqslant z \leqslant z^*,$$

where z^* is a root of the function $f(z)$, i.e. $f(z^*) = 0$. The second derivative $f''(z)$ is non-decreasing for $z_0 \leqslant z \leqslant z^*$.

Then

$$\frac{f''(z) f(z)}{[f'(z)]^2} \leqslant \frac{1}{2} \quad \text{for} \quad z_0 \leqslant z \leqslant z^*.$$

The following theorem contains sufficient conditions for the convergence of the process defined by the formula (2). These conditions assure also the existence of a solution of Eq. (1).

THEOREM. *Let us assume that* Eq. (1) *possesses a majorant* Eq. (3) *satisfying the following conditions*:

1) Eq. (3) *has a positive root.*

2) $f'(z) < 0$ *for* $z_0 \leqslant z \leqslant \frac{4}{3} z^*$,

where z^* is the smallest positive root of Eq. (3). *Under these assumptions the following assertion is true:*

The process defined by (4) *converges to the solution* z^* *of* Eq. (3). Eq. (1) *has a solution* x^* *and the sequence of the approximate solutions* x_n *defined by the formula* (2) *converges to it. The error estimate is given by the formula*

$$\|x^* - x_n\| \leqslant z^* - z_n. \tag{5}$$

Proof. First of all, we shall prove that the sequence of z_n is convergent. It follows from the hypotheses of the Theorem that this sequence is increasing. Thus, it is sufficient to show that this sequence is bounded. Using a simple argument and applying the Lemma one can show that

$$z_n \leqslant \frac{4}{3} z^*, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where z^* is the smallest positive root of Eq. (3), greater than z_0. Using the Lemma again we get

$$\frac{f(z_n)}{f'(z_n)} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, it follows from the conditions of the Theorem that

$$f(z_n) \to 0 \quad \text{as} \quad n \to \infty.$$

In virtue of the Lemma it follows from the assumptions of the Theorem that

$$\|x_1 - x_0\| \leqslant z_1 - z_0. \tag{7}$$

Further, we shall show the existence of the inverse Γ_1 of $P'(x_1)$ and that condition 2° is satisfied for x_1. Let us estimate the expression

$$\|I - \Gamma_0 P'(x_1)\| = \Big\| \int_{x_0}^{x_1} \Gamma_0 P''(x)\, dx \Big\| \leqslant B \int_{z_0}^{z_1} Q''(z)\, dz = 1 - \frac{Q'(z_1)}{Q'(z_0)} = q < 1. \tag{8}$$

We have

$$\Gamma_0 P'(x_1) = I - (I - \Gamma_0 P'(x_1)).$$

Thus, it follows the existence of the inverse of $P'(x_1)$, since $q < 1$. We have

$$\Gamma_1 = \{P'(x_0)(I - \Gamma_0 [P'(x_0) - P'(x_1)])\}^{-1}.$$

Hence, we get, by (8),

$$\|\Gamma_1\| \leqslant \|\Gamma_0\| \frac{1}{1 - \|I - \Gamma_0 P'(x_1)\|} \leqslant -Q'(z_1). \tag{9}$$

The existence of operator Q_1 follows from the Lemma. It remains to prove that condition 1° is fulfilled for x_1.

Using the analogue of Taylor's formula in the integral form we obtain, by (2),

$$P(x_1) = P(x_0) - P'(x_0) Q_0 \Gamma_0 P(x_0) + \\ + \frac{1}{2} P''(x_0)(Q_0 \Gamma_0 P(x_0))^2 + \frac{1}{6} \int_{x_0}^{x_1} P'''(\bar{x})(x_1 - \bar{x})^3\, d\bar{x}.$$

Put

$$A = \frac{1}{2}\Gamma_0 P''(x_0)\,\Gamma_0 P(x_0).$$

Then we get

$$Q_0 = (I - A)^{-1} = I + AQ_0.$$

Hence, we obtain

$$P'(x_0)\,Q_0\,\Gamma_0 P(x_0) = P(x_0) + \frac{1}{2}P''(x_0)\,\Gamma_0 P(x_0)\,Q_0\,\Gamma_0 P(x_0) \tag{10}$$

and, consequently,

$$Q_0\,\Gamma_0 P(x_0) = \Gamma_0 P(x_0) + y_0, \tag{11}$$

where

$$y_0 = \frac{1}{2}\Gamma_0 P''(x_0)\,\Gamma_0 P(x_0)\,Q_0\,\Gamma_0 P(x_0).$$

Further, we have, in virtue of (11),

$$P''(x_0)\,(Q_0\,\Gamma_0 P(x_0))^2 = P''(x_0)\,(\Gamma_0 P(x_0) + y_0)\,Q_0\,\Gamma_0 P(x_0) =$$

$$= P''(x_0)\,\Gamma_0 P(x_0)\,Q_0\,\Gamma_0 P(x_0) + P''(x_0)\,y_0\,Q_0\,\Gamma_0 P(x_0).$$

Hence, we get, by (10),

$$P(x_1) = \frac{1}{2}P''(x_0)\,y_0\,Q_0\,\Gamma_0 P(x_0) + \frac{1}{6}\int_{x_0}^{x_1} P'''(\bar{x})\,(x_1 - \bar{x})^3\,d\bar{x}. \tag{12}$$

On the other hand we have

$$f(z_1) = f(z_0) - \frac{f(z_0)}{1 - \frac{1}{2}a} + \frac{1}{2}\,\frac{f^2(z_0)}{[f'(z_0)]^2}\,\frac{f''(z_0)}{(1 - \frac{1}{2}a)^2} + \frac{1}{6}\int_{z_0}^{z_1} f'''(z)\,(z_1 - z)^3\,dz.$$

Hence, we obtain

$$f(z_1) = \frac{1}{4}\,\frac{a_2}{(1 - \frac{1}{2}a)^2}\,f(z_0) + \frac{1}{6}\int_{z_0}^{z_1} f'''(z)\,(z_1 - z)^3\,dz, \tag{13}$$

where

$$a = \frac{f''(z_0)\,f(z_0)}{[f'(z_0)]^2}.$$

It follows from (7), (12) and (13), using conditions 1°—4°, that

$$\|P(x_1)\| \leqslant f(z_1).$$

Therefrom we infer that

$$\|x_2 - x_1\| \leqslant z_2 - z_1.$$

Thus, one can show by induction that conditions 1°—4° are satified for $x = x_n$, $n = 1, 2, \ldots$. As a consequence we get

$$\|x_{n+p} - x_n\| \leqslant z_{n+p} - z_n$$

for any positive integers p and n. Hence, it follows that condition (5) is satisfied and the sequence of the approximate solutions x_n defined by the formula (2) converges toward an element x^* of the space X. Since

$$\|P(x_n)\| \leqslant f(x_n),$$

we infer that

$$P(x_n) \to 0 \quad \text{as} \quad n \to \infty$$

and by the continuity of $P(x)$ we conclude that x^* is a solution of Eq. (1).

Remark 1. Conditions 1°—4° can be replaced by the following ones:

$$\left.\begin{array}{ll} 1^\circ & \|\Gamma_0 P(x_0)\| \leqslant B_0 f(z_0); \\ 2^\circ & \|\Gamma_0 P''(x)\| \leqslant B_0 f''(z) \quad \text{if} \\ 3^\circ & \|\Gamma_0 P'''(x)\| \leqslant B_0 f''(z) \quad \text{if} \end{array}\right\} \|x - x_0\| \leqslant z - z_0 \leqslant z' - z_0;$$

Remark 2. Condition 2° can be replaced by the following one:

$$f'(z_n) < 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Remark 3. In virtue of our Lemma condition 1) of the theorem in [2] can be omitted. The same is true for the theorem in [1].

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MATHEMATICS

An Iterative Method for Eigenvalues and Eigenvectors of Matrices

by

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Presented by W. ORLICZ on July 1, 1961

This paper contains an iterative method*) for computing eigenvalues and eigenvectors of real symmetric matrices. This method is based on a general method of finding zero-elements of a non-linear functional in a Banach space (see [1]). However, the proof of the convergence of the method in question requires a different argument. The advantage of this method is that, starting from arbitrary but different vectors, we obtain in general different eigenvalues and the corresponding eigenvectors. The method is suitable for programming on a computer.

The first experiment with this method has been carried out on the computer IBM 704 at the Computing Center of the New York University**). The results of the calculation were quite satisfactory. A further experiment has been carried out on the computer XYZ ***). Starting from the base vectors $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, ..., different general eigenvalues were obtained. The speed of the convergence was satisfactory even in the case when the ratio of the largest eigenvalue calculated to the next one was 10:9. Of course, the power method cannot be recommended in this case, because of too slow convergence.

Let A be a real and symmetric matrix of order N. Thus, A defines a linear operator in the Euclidean space R_N. The problem is to find eigenvalues λ and corresponding eigenvectors $x \in R_N$ satisfying the equation

$$Ax = \lambda x, \quad x \neq 0. \tag{1}$$

It follows from (1)

$$\lambda x = \frac{(Ax, x)}{\|x\|^2}. \tag{2}$$

Hence, by (2), we get instead of (1) the following equation

$$Ax = \lambda(x)\, x, \quad x \neq 0.$$

*) This method has been first formulated in my notes to the lectures given in 1959 at the California Institute of Technology.

**) Program and computations were carried out by B. G. Weitzenhoffer.

***) Program and computations were carried out by W. Dubnicki, the computer XYZ is in ZAM, Warsaw.

Instead of this operator equation we shall solve the following functional equation

$$\|Ax - \lambda(x)\, x\|^2 = 0$$

or, equivalently,

$$F(x) = \|x\|^2 \|Ax\|^2 - (Ax, x)^2 = 0, \qquad x \neq 0. \tag{3}$$

Thus, we see that our eigenvector problem is equivalent to that of finding the solutions of Eq. (3).

But this equation means that the value of cosine of the angle between the eigenvector x and its image Ax is equal to 1 or -1. For this reason we shall say that Eq. (3) is the cosine equation for the matrix A. Consequently, the eigenvector or the eigenvalue problem for the matrix A is equivalent to that of finding the solutions of the cosine equation for the matrix A. But the cosine equation (3) is a non-linear functional equation with a functional being continuously differentiable in the sense of Fréchet. A method of solving such non-linear functional equations was given in [1] and [2].

We shall now apply this method to our eigenproblem. Then the sequence of approximate solutions x_n of Eq. (3) is defined as follows:

$$x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|^2} F'(x_n), \qquad n = 0, 1, 2, \ldots . \tag{4}$$

First of all, we shall calculate the Fréchet derivative $F'(x)$ of $F(x)$ in (4). Using the abstract analogues of the differential chain rules, we obtain

$$\frac{1}{2} F'(x) = y = \|Ax\|^2 x + \|x\|^2 A^2 x - 2(Ax, x)\, Ax. \tag{5}$$

Then we get instead of (4)

$$x_{n+1} = x_n - \frac{\|x_n\|^2 \|Ax_n\|^2 - (Ax_n, x_n)^2}{2\|y_n\|^2} y_n, \tag{6}$$

where y_n is defined by the formula in (5) in which x should be replaced by x_n.

We can now establish the following

THEOREM. *Let x_0 be an arbitrary vector such that condition* (14) *is not satisfied. Then the sequence of numbers $\dfrac{(Ax_n, x_n)}{\|x_n\|^2}$ converges to an eigenvalue λ of the matrix A. For sufficiently large n we have the following error estimate*

$$\left| \lambda - \frac{(Ax_n, x_n)}{\|x_n\|^2} \right| \leqslant \frac{\sqrt{2F(x_n)}}{\|x_n\|^2}. \tag{7}$$

If λ is a simple eigenvalue, then the sequence of approximate eigenvectors x_n converges towards the eigenvector x^* corresponding to λ and we have

$$\left\| \frac{(x_n, x^*)}{\|x^*\|^2} x^* - x_n \right\| \leqslant \frac{\sqrt{2F(x_n)}}{c\,\|x_n\|} \tag{8}$$

for sufficiently large n, where c is the distance from λ to the nearest eigenvalue.

If the eigenvalue λ has the multiplicity greater than one, then for sufficiently large n there exist an eigenvector x_n^* corresponding to λ such that

$$\|x_n^* - x_n\| \leqslant \frac{\sqrt{2F(x_n)}}{c\,\|x_n\|}. \tag{9}$$

Proof. It is easy to see, by (4) and (5), that

$$(y, x) = 2F(x). \tag{10}$$

We have, by (6) and (10),

$$\|x_{n+1}\|^2 = \|x_n\|^2 - \frac{7}{4}\,\frac{F^2(x_n)}{\|y_n\|^2}. \tag{11}$$

It follows from (11) that the sequence of $\|x_n\|^2$ is convergent, since it is decreasing and bounded. Hence, we get

$$F(x_n) \to 0 \quad \text{as} \quad n \to \infty, \tag{12}$$

since the sequence of $\|y_n\|^2$ is bounded and

$$\frac{F^2(x_n)}{\|y_n\|^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Relation (11) yields

$$\|x_{n+1}\|^2 = \|x_0\|^2 - \frac{7}{4}\sum_{k=0}^{n}\frac{F^2(x_k)}{\|y_k\|^2}.$$

Thus, the series

$$\sum_{k=0}^{\infty}\frac{F^2(x_k)}{\|y_k\|^2} < +\infty \tag{13}$$

and

$$x_n \to 0 \quad \text{as} \quad n \to \infty,$$

if and only if

$$\|x_0\|^2 = \frac{7}{4}\sum_{k=0}^{\infty}\frac{F^2(x_k)}{\|y_k\|^2}. \tag{14}$$

We shall now establish some relations needed for the proof of the Theorem.

Let $e_1, e_2, \ldots, e_N$ be the normed eigenvectors of the matrix A. They form an orthogonal system and any vector x of R_N can be represented in the form

$$x = \sum_{i=1}^{N}\xi_i e_i. \tag{15}$$

Thus, we get the following expression for y in (5)

$$y = \sum_{i=1}^{N}\|Ax - \lambda_i x\|^2\,\xi_i e_i, \tag{16}$$

where λ_i are the eigenvalues corresponding to e_i. In virtue of (15) and (16) we obtain

$$(y, x) = \sum_{i=1}^{N} \|Ax - \lambda_i x\|^2 \xi_i^2 \tag{17}$$

and

$$(Ay, x) = \sum_{i=1}^{N} \|Ax - \lambda_i x\|^2 \lambda_i \xi_i^2. \tag{18}$$

In the same way we get the following expression for $F(x)$ in (3)

$$F(x) = \|x\|^2 \sum_{i=1}^{N} \left(\lambda_i - \frac{(Ax, x)}{\|x\|^2}\right)^2 \xi_i^2 . \tag{19}$$

We can now prove the convergence of the sequence of numbers (Ax_n, x_n). We have, by (6),

$$(Ax_{n+1}\, x_{n+1}) = \left(Ax_n - \frac{F(x_n)}{2\|y_n\|^2} Ay_n, \quad x_n - \frac{F(x_n)}{2\|y_n\|^2} y_n\right).$$

Hence, we get

$$(Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \frac{F^2(x_n)}{4\|y_n\|^2}\left[\frac{(Ay_n, y_n)}{\|y_n\|^2} - 4\frac{(Ay_n, x_n)}{F(x_n)}\right]. \tag{20}$$

We shall show that the expression in square brackets in (20) is bounded. In fact, we have

$$\frac{|(Ay_n, y_n)|}{\|y_n\|^2} \leqslant M \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{21}$$

where

$$M = \max_{1 \leqslant i \leqslant N} |\lambda_i| .$$

Further, by (18) and (17), we have

$$|(Ay, x)| \leqslant M(y, x) = 2MF(x). \tag{22}$$

Hence, we get, by (22),

$$\frac{|(Ay_n, x_n)|}{F(x_n)} \leqslant 2M \quad \text{for} \quad n = 0, 1, 2, \ldots . \tag{23}$$

If $\|y_n\|$ in (21) or $F(x_n)$ in (23) vanish, then, of course, x_n is an eigenvector. Since the series in (13) is convergent, so is the series

$$\sum_{n=0}^{\infty} (Ax_{n+1}, x_{n+1}) - (Ax_n, x_n) = \lim_{k \to \infty} (Ax_k, x_k) - (Ax_0, x_0).$$

Since the sequence of $\|x_n\|^2$ is convergent, the proof of the convergence of the sequence $\left\{\frac{(Ax_n, x_n)}{\|x_n\|^2}\right\}$ is accomplished. Denote by λ its limit. It is easy to see that λ is an eigenvalue of A. For we have

$$F(x_n) = \|x_n\|^2 \left\| Ax_n - \frac{(Ax_n, x_n)}{\|x_n\|^2} x_n \right\|^2 \to 0 \quad \text{as} \quad n \to \infty .$$

Since the sequence of $\|x_n\|$ does not converge towards zero, there is a subsequence of $\{x_n\}$ convergent to some element x^* and

$$\|Ax^* - \lambda x^*\| = 0 .$$

It is evident that if the eigenvalue λ is simple, then we have

$$x_n \to x^* \quad \text{as} \quad n \to \infty .$$

We shall now prove the error estimate in (9). Suppose that the multiplicity of the eigenvalue $\lambda = \lambda_1$ is $r < N$. Let $e_1, e_2, \ldots, e_r$ be the eigenvectors corresponding to λ. Then we have, by (19),

$$F(x_n) \geqslant \frac{c^2}{2} \|x_n\|^2 \sum_{i=r+1}^{N} \xi_{in}^2 , \tag{24}$$

where

$$x_n = \sum_{i=1}^{N} \xi_{in} e_i ,$$

if n is sufficiently large, since in this case the following inequalities hold

$$\left| \lambda_i - \frac{(Ax_n, x_n)}{\|x_n\|^2} \right| \geqslant \frac{c}{2} \quad \text{for} \quad i = r+1, r+2, \ldots, N.$$

Put

$$x_n^* = \sum_{i=1}^{r} \xi_{in} e_i .$$

Of course, x_n^* is an eigenvector corresponding to λ and

$$\|x_n^* - x_n\|^2 = \sum_{i=r+1}^{N} \xi_{in}^2 . \tag{25}$$

Hence, in virtue of (24), we get relation (9).

It is obvious that the same is true if the eigenvalue λ is simple, i.e. $r = 1$, and this is equivalent to the relation (8).

In the same way we obtain the error estimate for the approximate eigenvalue. Thus, we obtain

$$\sum_{i=1}^{r} \xi_{in}^2 = \|x_n\|^2 - \sum_{i=r+1}^{N} \xi_{in}^2 .$$

We have, by (24),

$$\sum_{i=r+1}^{N} \xi_{in}^2 \to 0 \quad \text{as} \quad n \to \infty .$$

Hence, for sufficiently large n we obtain

$$\sum_{i=1}^{r} \xi_{in}^2 \geqslant \frac{\|x_n\|^2}{2}. \tag{26}$$

Using the relation (19) we get, by (26),

$$F(x_n) \geqslant \frac{1}{2}\|x_n\|^4 \left(\lambda - \frac{(Ax_n, x_n)}{\|x_n\|^2}\right)^2,$$

whence follows relation (7).

Remark. It is rather unlikely for condition (14) to be satisfied in practice. Thus, process (6) gives practically always an approximate eigenvalue and a corresponding approximate eigenvector.

The same method can be applied to selfadjoint linear operators in Hilbert space.

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MATHEMATICS

A General Method of Steepest Ortho-descent

by

M. ALTMAN

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1. Paper [1] contains an iterative method of solving linear equations in Hilbert space. The characteristic feature of this method is that the corrections improving the approximate solutions obtained at each iteration step are orthogonal to the vector b of Eq. (1). It is shown in [1] that the speed of the convergence depends then essentially on the vector b. This fact makes it possible to introduce an arbitrary vector as a parameter. Thus, changing the parameter vector we change the speed of the convergence at the same time. From this point of view the method presented in [1] gives a kind of over-relaxation, where the numerical parameter is replaced by an arbitrary vector.

In this paper we give a generalization of the main idea contained in [1]. This generalization means that the correction vectors improving the approximate solutions are orthogonal to b and to a subspace as well. This subspace can be introduced in many ways. The speed of the convergence depends now on both factors: on b and on the choice of the subspace. Thus, we obtain a kind of over-relaxation depending on a subspace as a parameter. The general method of steepest orthogonal descent presented here is faster convergent than the method in [1] and by a suitable choice of the subspace it can be much faster convergent than the method of steepest descent investigated by L. V. Kantorovitch [2]. An application to the solution of a system of linear algebraic equations is also given.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \qquad x, b \in H. \tag{1}$$

We shall assume operator A to be selfadjoint and positive definite, or, more precisely,

$$A^* = A,$$

where A^* is the adjoint of A, and

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M < \infty$, m and M being the minimum and maximum eigenvalues of A, respectively.

Without loss of generality one can suppose that $\|b\| = 1$.

Let now L be an arbitrary subspace of H orthogonal to b and to Ab, i.e.

$$b \perp L \quad \text{and} \quad Ab \perp L.$$

Let us put

$$Ł = A(L),$$

i.e. $Ł$ is the image of the subspace L. Then the subspace $Ł$ is orthogonal to b.

Let Z be the complementary subspace of H orthogonal to b and to $Ł$, i.e.

$$Z \perp b \quad \text{and} \quad Z \perp Ł.$$

Denote by $P_b x$ the orthogonal projection of $x \in H$ onto the subspace of H spanned by the vector b.

Denote by $P_Ł x$ the orthogonal projection of x onto the subspace $Ł$ of H, and by $P_Z x$ the orthogonal projection of x onto the subspace Z of H.

We have

$$P_Z x = x - P_b x - P_Ł x. \tag{2}$$

Let us now define the following linear operator R

$$z = Rx = P_Z Ax = Ax - P_b Ax - P_Ł Ax.$$

Hence, we get the following expression for Rx

$$z = Rx = Ax - (Ax, b)\, b - A\tilde{x}, \tag{3}$$

where

$$A\tilde{x} = P_Ł Ax.$$

Consider now the following homogeneous equation

$$Rx = 0, \quad x \neq 0, \tag{4}$$

where operator R is defined in (3).

We shall now define an iterative process of solving Eq. (4). Let x_0 be an arbitrary element of H satisfying the following condition

$$(x_0, b) \neq 0. \tag{5}$$

Put

$$x_{n+1} = x_n - a_n z_n, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where

$$z_n = Rx_n.$$

Thus, we have

$$z_{n+1} = z_n - a_n Rz_n. \tag{7}$$

To determine the coefficients a_n in (6) consider the quadratic functional

$$F(z) = (R^{-1} z, z), \tag{8}$$

defined for all z of Z. To show that this definition makes sense we have to prove the existence of the inverse of the operator R with domain and range in the subspace Z. But this is easy to verify, since the following relation is true

$$(Rz, z) = (Az, z) \tag{9}$$

for any z of Z.

Thus, we see that operator R is positive defined and selfadjoint in the subspace Z. Hence, it follows the existence of the inverse R^{-1} defined on Z.

Denote by m_Z, M_Z the minimum and maximum eigenvalues of R in Z.

Thus, we have, by (9)

$$m_Z(z, z) \leqslant (Rz, z) \leqslant M_Z(z, z) \text{ and } m \leqslant M_Z, M_Z \leqslant M. \tag{10}$$

We shall now choose α_n in (6) and in (7) so as to minimize $F(z_{n+1})$, where $F(z)$ is defined in (8). Thus, we obtain

$$\alpha_n = \frac{(z_n, z_n)}{(Rz_n, z_n)} = \frac{(z_n, z_n)}{(Az_n, z_n)}, \tag{10'}$$

and the iterative process defined by (6) and (7) is of the form

$$x_{n+1} = x_n - \frac{(z_n, z_n)}{(Rz_n, z_n)} z_n, \tag{11}$$

$$z_{n+1} = z_n - \frac{(z_n, z_n)}{(Rz_n, z_n)} Rz_n, \tag{12}$$

where operator R is defined in (3). We shall now show that the sequence of z_n converges to the zero element of H. We have

$$F(z_{n+1}) = F(z_n) - \frac{(z_n, z_n)^2}{(Rz_n, z_n)}.$$

Hence, it follows that

$$\frac{F(z_{n+1})}{F(z_n)} = 1 - \frac{(z_n, z_n)^2}{(R^{-1} z_n, z_n)(Rz_n, z_n)}.$$

Since (see [1])

$$\frac{(z_n, z_n)^2}{(R^{-1} z_n, z_n)(Rz_n, z_n)} \geqslant \frac{4M_Z m_Z}{(M_Z + m_Z)^2},$$

we obtain

$$\frac{F(z_{n+1})}{F(z_n)} \leqslant \left(\frac{M_Z - m_Z}{M_Z + m_Z}\right)^2. \tag{13}$$

It follows from (10) that

$$M_Z^{-1}(z, z) \leqslant (R^{-1} z, z) \leqslant m_Z^{-1}(z, z). \tag{14}$$

Relations (13) and (14) yield

$$\|z_n\| \leqslant \sqrt{M_Z}\, \sqrt{F(z_0)}\, q_Z^n, \tag{15}$$

where

$$q_Z = \frac{M_Z - m_Z}{M_Z + m_Z}.$$

Thus, the sequence of z_n converges to the zero element of H at least as fast as a geometric progression with ratio q_Z.

It follows from (6) that

$$x_n = x_0 - a_0 z_0 - \ldots - a_{n-1} z_{n-1}, \tag{16}$$

where $z_i \in Z$ for $i = 0, 1, 2, \ldots$.

Hence, we infer, by (15) and (10′), that the sequence of x_n is convergent. Consequently, we have $x_n \to x = x_0 - z$ as $n \to \infty$, where z is an element of Z. It is easy to see that $x = x_0 - z$ is a solution of Eq. (4). In fact, we have

$$z_n = Rx_n \to Rx.$$

Since the sequence of z_n converges to 0, we get $Rx = 0$. It remains to show that x is not a trivial solution of Eq. (4). We have, by (16) and (15),

$$(x_n, b) = (x_0, b) \neq 0,$$

since elements z_i are orthogonal to b.

Hence, we get

$$\|x_n\| \geqslant |(x_0, b)| > 0. \tag{17}$$

Now, if for x_n defined by relation (11) the norm of z_n is sufficiently small, then we calculate the approximate solution y_n of Eq. (1) by the following formula:

$$y_n = \frac{1}{\beta_n}(x_n - \tilde{x}_n), \tag{18}$$

where

$$\beta_n = (Ax_n, b) \tag{19}$$

and, where $\tilde{x}_n$ is defined by relation

$$P_L Ax_n = A\tilde{x}_n. \tag{20}$$

We shall show that

$$\beta_n \to \beta \neq 0 \quad \text{as} \quad n \to \infty,$$

where

$$\beta = (Ax, b) \tag{21}$$

and, where x is the solution of Eq. (4) obtained above as a limit of the sequence of x_n. We have, by (3),

$$z_n = Rx_n = A(x_n - \tilde{x}_n) - \beta_n b, \tag{22}$$

where $\tilde{x}_n$ and β_n are the same as in (19), (20). It follows from (20) that

$$\tilde{x}_n \to \tilde{x} \quad \text{as} \quad n \to \infty.$$

Since $A\tilde{x}_n$ are in $\mathcal{L}$, so is $A\tilde{x}$, i.e.

$$A\tilde{x} \in \mathcal{L}$$

or, equivalently, $\tilde{x} \in L$.

Thus, we obtain from (22) the following equation being equivalent to Eq. (4)

$$Rx = A(x - \tilde{x}) - \beta b = 0, \tag{23}$$

where β is defined in (21).

Suppose for a moment that $\beta = 0$. Hence, it follows from (23) that $x = \tilde{x}$ and, consequently,

$$x = x_0 + z = \tilde{x},$$

where $z \in Z$ and $\tilde{x} \in L$.

Hence, we get the following relation

$$(x_0, b) = 0,$$

which is a contradiction to (5). Hence, we obtain that

$$x \neq \tilde{x},$$

where x is the solution of Eq. (4) and $A\tilde{x} = P_{\mathcal{L}} Ax$. It is easy to see that for sufficiently large n we have

$$x_n \neq \tilde{x}_n,$$

where $A\tilde{x}_n = P_{\mathcal{L}} Ax_n$.

We shall now give an error estimate for the approximate solutions y_n of Eq. (1). We have, by (22), (14), (15) and (19),

$$\|y_n - x^*\| \leqslant \frac{\sqrt{M_Z}\sqrt{F(z_0)}}{m\,|(Ax_n, b)|}\, q_Z^n \leqslant \frac{M_Z \|z_0\|}{m\sqrt{m_Z}\,|(Ax_n, b)|}\, q_Z^n, \tag{24}$$

where x^* is the solution of Eq. (1).

According to the error estimate given by Kantorovitch [2] the method of steepest descent is at least as fast as a geometric progression with ratio

$$q = \frac{M - m}{M + m}.$$

Thus, we see that the general method of steepest orthogonal descent given above can be much faster than the method of steepest descent, provided that a suitable subspace L is chosen.

One can obtain various methods by specializing the subspace L. In particular, if we replace L by the subspace consisting of the zero element only, then we obtain the method of steepest orthogonal descent given in [1].

2. In the same way as in [1] one can introduce an additional parameter vector.

Let y be an arbitrary fixed vector of H. If x^* is the solution of Eq. (1), then, obviously, we have

$$A(x^* + y) = Ay + b.$$

Putting

$$b' = \frac{Ay+b}{\|Ay+b\|}, \qquad x = \frac{x^*+y}{\|Ay+b\|}$$

we obtain the following equation

(25) $$Ax = b'$$

instead of Eq. (1). We can now apply the method, described in par. 1 to solve Eq. (25). The chosen subspace L should be orthogonal to b' and Ab'. The starting vector x_0 should satisfy the following condition

$$(x_0, Ay+b) \neq 0,$$

which replaces that in (5). We define the operator R_y replacing in (3) b by b' and we obtain the following equation instead of (4).

(26) $$R_y x = Ax - \frac{(Ax, Ay+b)}{\|Ay+b\|^2}(Ay+b) - A\tilde{x} = 0,$$

where

$$A\tilde{x} = P_{\mathfrak{L}} Ax.$$

Replacing in (11), (12) R by R_y we obtain the following formulas for the approximate solutions of Eq. (26).

$$x_{n+1} = x_n - a_n z_n,$$

$$z_{n+1} = z_n - a_n R_y z_n,$$

$$a_n = \frac{(z_n, z_n)}{(R_y z_n, z_n)} = \frac{(z_n, z_n)}{(Az_n, z_n)}.$$

Thus, we have obtained a sequence of z_n convergent to zero. If the norm of z_n is sufficiently small, we take

$$y_n = \frac{\|Ay+b\|^2}{(Ax_n, Ay+b)}(x_n - \tilde{x}_n) - y$$

as the approximate solution of Eq. (1). If Z_y is the complementary subspace of H orthogonal to b' and to $\mathfrak{L} = A(L)$, then the operator R_y with domain and range in Z_y is selfadjoint and positive definite having m_y, M_y as the minimum and masimum eigenvalues, respectively. To obtain the error estimate for y_n we should replace in (24) m_Z, M_Z by m_y, M_y, respectively.

Remark 1. One can apply the Tchebyshev-polynomials to accelerate the convergence of the sequence z_n.

Remark 2. Suppose that operator A is defined by the matrix

$$A = (a_{ik}), \qquad i, k = 1, 2, \ldots, N.$$

Let $l_1, l_2, \ldots, l_s$ be linearly independent vectors orthogonal to b and to Ab. Orthogonalizing Al_1, $Al_2, \ldots, Al_s$ we obtain an orthonormal system $u_1, u_2, \ldots, u_s$,

($s < N$) forming a basis in the subspace $\mathcal{L}$ orthogonal to b. Then operator R can be obtained in a matrix form as follows. Put

$$b = (b_1, b_2, \ldots, b_N), \qquad c_k = (a_k, b), \qquad a_k = (a_{1k}, a_{2k}, \ldots, a_{Nk}),$$

$$l_j = (l_{1j}, l_{2j}, \ldots, l_{Nj}), \qquad d_{ki} = (a_k, l_j) \qquad j = 1, 2, \ldots, s.$$

Then we have

$$R = (r_{ik}) \qquad i, k = 1, 2, \ldots, N,$$

where

$$r_{ik} = a_{ik} - c_k b_i - d_{k1} l_{i1} - d_{k2} l_{i2} - \ldots - d_{ks} l_{is}.$$

Other generalizations of the method of steepest orthogonal descent will be given later.

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MATHEMATICS

A Gradient Relaxation Method for Linear Algebraic Equations

by

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Presented by W. ORLICZ on July 4, 1961

1. This paper contains an iterative method of solving systems of linear algebraic equations. The method described below is closely connected with the method of steepest descent investigated by L. V. Kantorovitch [1]. The difference between these methods can be explained as follows. In the method of steepest descent one moves in the gradient direction so as to minimize the error measure at each iteration step. However, the convergence obtained in this way is usually too slow. In the method presented here one improves the direction by a relaxation of the gradient itself so as to reduce the error measure still more at each iteration step. Thus, one obtains a faster convergence.

We consider here two variants of the method. The first variant consists in using one gradient relaxation at each iteration step. The second variant is a generalization of the first one and is obtained by repeating the gradient relaxation several times at each iteration step. Concerning programming on computer there is a difference between these variants. While the first variant does not need any additional commands, the second one requires to choose from two vectors a pair of co-ordinates with the same index but different signs.

Let A be a real symmetric and positive definite matrix

$$A = (a_{ik}), \qquad i, k = 1, 2, \ldots, N.$$

Consider the linear equation

$$Ax = b, \tag{1}$$

where x and b are vectors in the N-dimensional Euclidean space. Denote by x^* the solution of Eq. (1). Put

$$r = Ax - b$$

and consider the quadratic functional

$$F(x) = (r, A^{-1} r) = (Ax - b, x - x^*).$$

$F(x)$ is to be considered as an error measure for the approximate solution of Eq. (1), and $2r$ is the gradient of the functional $F(x)$.

Let x_0 be an arbitrary vector. Then the method of steepest descent [1] is defined as follows.

$$x_{n+1} = x_n - \frac{(r_n, r_n)}{(Ar_n, r_n)} r_n, \tag{2}$$

$$r_{n+1} = r_n - \frac{(r_n, r_n)}{(Ar_n, r_n)} Ar_n, \tag{3}$$

where

$$r_n = Ax_n - b.$$

We have then

$$F(x_{n+1}) = F(x_n) - \frac{(r_n, r_n)^2}{(Ar_n, r_n)}. \tag{4}$$

Let us take another vector y_n and consider the method defined as follows

$$x_{n+1} = x_n - a_n y_n,$$

where a_n is so chosen as to minimize the error measure $F(x_{n+1})$. Then we get

$$a_n = \frac{(r_n, y_n)}{(Ay_n, y_n)}.$$

Thus, we obtain

$$x_{n+1} = x_n - \frac{(r_n, y_n)}{(Ay_n, y_n)} y_n, \tag{5}$$

$$r_{n+1} = r_n - \frac{(r_n, y_n)}{(Ay_n, y_n)} Ay_n. \tag{6}$$

Hence, it follows that

$$F(x_{n+1}) = F(x_n) - \frac{(r_n, y_n)^2}{(Ay_n, y_n)}. \tag{7}$$

We shall now choose the vector y_n. Let $u_1, u_2, \ldots, u_N$ be an arbitrary basis. Suppose that x_n and r_n are given. Then we calculate Ar_n. Let k be an index such that

$$(r_n, u_k)(Ar_n, u_k) < 0. \tag{8}$$

Then we put

$$y_n = r_n - \frac{(Ar_n, u_k)}{(Au_k, u_k)} u_k \tag{9}$$

and obtain

$$Ay_n = Ar_n - \frac{(Ar_n, u_k)}{(Au_k, u_k)} Au_k. \tag{10}$$

We have

$$(Ay_n, u_k) = (y_n, Au_k) = 0.$$

It follows from (9) that

$$(r_n, y_n) = (r_n, r_n) - \frac{(r_n, u_k)(Ar_n, u_k)}{(Au_k, u_k)}. \tag{11}$$

Relations (9) and (10) yield

$$(Ay_n, y_n) = (Ar_n, r_n) - \frac{(Ar_n, u_k)^2}{(Au_k, u_k)}. \tag{12}$$

Thus, we get for relation (7), by (11) and (12),

$$F(x_{n+1}) = F(x_n) - \frac{\left[(r_n, r_n) - \frac{(r_n, u_k)\ (Ar_n, u_k)}{(Au_k, u_k)}\right]^2}{(Ar_n, r_n) - \frac{(Ar_n, u_k)^2}{(Au_k, u_k)}}. \tag{13}$$

A comparison of relations (7) and (13) shows, by (8), that the error measure is still more diminishing for the method defined by (5), (6) and, consequently, the convergence of this method is faster than the convergence of the method of steepest descent.

Putting

$$u_1 = (1, 0, ..., 0), \quad u_2 = (0, 1, 0, ..., 0), ..., \quad u_N = (0, ..., 0, 1) \tag{14}$$

we have in relations (9)—(12)

$$(Au_k, u_k) = a_{kk}. \tag{14'}$$

Hence, it follows that we can simplify the calculations dividing each equation of the system (1) by the diagonal element of the matrix A. Thus, we obtain for relations (5), (6), (9) and (10)

$$x_{n+1} = x_n - \frac{(r_n, r_n) - (r_n, u_k)\ (Ar_n, u_k)}{(Ar_n, r_n) - (Ar_n, u_k)^2} y_n, \tag{15}$$

$$r_{n+1} = r_n - \frac{(r_n, r_n) - (r_n, u_k)\ (Ar_n, u_k)}{(Ar_n, r_n) - (Ar_n, u_k)^2} Ay_n, \tag{16}$$

$$y_n = r_n - (Ar_n, u_k)\, u_k, \tag{17}$$

$$Ay_n = Ar_n - (Ar_n, u_k)\, Au_k. \tag{18}$$

It is easy to see that using the computation scheme defined by relations (15)—(18) we do not need to compare the signs of the corresponding coordinates of the vectors r_n and Ar_n in order to choose k satisfying condition (8).

2. We shall now consider the second variant of the method which is a generalization of the first one. Let x_0 be an arbitrary vector. Suppose that x_n and r_n are already given. Then we compute Ar_n and we carry out the first relaxation step as in par. 1.

In other words, let us put

$$y_n = r_n - \frac{(Ar_n, u_{k_1})}{(Au_{k_1}, u_{k_1})} u_{k_1}, \tag{19}$$

$$Ay_n = Ar_n - \frac{(Ar_n, u_{k_1})}{(Au_{k_1}, u_{k_1})} Au_{k_1}, \tag{20}$$

where k_1 is so chosen as to satisfy condition (8) replacing there k by k_1. We repeat now this relaxation step and put

$$y_{n+1} = y_n - \frac{(Ay_n, u_{k_2})}{(Au_{k_2}, u_{k_2})} u_{k_2}, \tag{21}$$

$$Ay_{n+1} = Ay_n - \frac{(Ay_n, u_{k_2})}{(Au_{k_2}, u_{k_2})} Au_{k_2}, \tag{22}$$

where k_2 is so chosen as to satisfy the following condition

$$(r_n, u_{k_2})(Ay_n, u_{k_2}) < 0.$$

Suppose that the last relaxation step gives y_{n+s-1}, i.e. after s relaxation steps we obtain

$$y_{n+s-1} = y_{n+s-2} - \frac{(Ay_{n+s-2}, u_{k_s})}{(Au_{k_s}, u_{k_s})} u_{k_s}, \tag{23}$$

$$Ay_{n+s-1} = Ay_{n+s-2} - \frac{(Ay_{n+s-2}, u_{k_s})}{(Au_{k_s}, u_{k_s})} Au_{k_s}, \tag{24}$$

where k_s is so chosen as to satisfy the following condition

$$(r_n, u_{k_s})(Ay_{n+s-2}, u_{k_s}) < 0. \tag{25}$$

Then we put

$$z_n = y_{n+s-1}, \qquad Az_M = Ay_{n+s-1},$$

$$x_{n+1} = x_n - \frac{(r_n, z_n)}{(Az_n, z_n)} z_n, \tag{26}$$

$$r_{n+1} = r_n - \frac{(r_n, z_n)}{(Az_n, z_n)} Az_n. \tag{27}$$

For the error measure we have relation (7) in which y_n should be replaced by z_n.

Let us see how this error measure diminishes. By a simple calculation we get

$$(r_n, z_n) = (r_n, r_n) - \frac{(r_n, u_{k_1})(Ar_n, u_{k_1})}{(Au_{k_1}, u_{k_1})} - \frac{(r_n, u_{k_2})(Ay_n, u_{k_2})}{(Au_{k_2}, u_{k_2})} - \frac{(r_n, u_{k_s})(Ay_{n+s-2}, u_{k_s})}{(Au_{k_s}, u_{k_s})}$$

and

$$(Az_n, z_n) = (Ar_n, r_n) - \frac{(Ar_n, u_{k_1})^2}{(Au_{k_1}, u_{k_1})} - \frac{(Ay_n, u_{k_2})^2}{(Au_{k_2}, u_{k_2})} - \frac{(Ay_{n+s-2}, u_{k_s})^2}{(Au_{k_s}, u_{k_s})}.$$

Since

$$F(x_{n+1}) = F(x_n) - \frac{(r_n, z_n)^2}{(Az_n, z_n)}, \tag{28}$$

we see that each relaxation step causes an increase of the numerator and a decrease of the denominator of the ratio in (28). Relations (26) and (27) determine each basic

iteration step to reduce the error measure, or, equivalently, to improve the approximate solution of Eq. (1). On the other hand, relations (19)—(25) determine the relaxation steps for each basic iteration to get a better direction of the vector z_n which gives a steeper descent for the error measure. There is no restriction how many relaxation steps should be carried out for each basic iteration unless there is no such k_s satisfying condition (25). To start the next basic iteration we should calculate Ar_{n+1} and we continue the process described above starting with a relaxation step.

If the vectors $u_1, u_2, \ldots, u_N$ are the same as in (14), then the corresponding expressions in relations (19)—(24) can be simplified by using (14′). Dividing each equation of the system (1) by the diagonal element of the matrix A we obtain a further simplification of calculations.

The number of relaxation steps can be different for each basic iteration and can be regulated depending on the results obtained during the computation process. An extension of this method to operator equations in Hilbert space can be given.

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MATHEMATICS

Some properties of (K, φ)-summability

by

R. TABERSKI

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1. In this paper the function $\varphi(x)$ is defined, Riemann integrable in the interval $\langle 0,1\rangle$ and

$$\int_0^1 \varphi^2(x)\,dx > 0, \qquad \sum_{\nu=0}^{n+2} \varphi^2\left(\frac{\nu}{n+2}\right) > 0 \qquad (n = 0, 1, 2, \ldots).$$

Further restrictions on φ are imposed in particular statements. We write

$$A_k^{(n)} = \sum_{\nu=0}^{n-k+2} \varphi\left(\frac{\nu}{n+2}\right)\varphi\left(\frac{\nu+k}{n+2}\right), \qquad A_n = A_0^{(n)} \qquad (k = 0, 1, \ldots, n;\ n = 0, 1, \ldots).$$

Evidently,

$$\frac{1}{n} A_n = \int_0^1 \varphi^2(x)\,dx + o(1) \qquad (\text{as } n \to \infty). \tag{1}$$

Let

$$\varrho_k^{(n)} = A_k^{(n)}/A_n \quad \text{for} \quad k \leqslant n, \qquad \varrho_k^{(n)} = 0 \quad \text{for} \quad k > n,$$

and

$$\lambda_k^{(n)} = \varrho_k^{(n)} - \varrho_{k+1}^{(n)} \quad \text{for} \quad k < n, \qquad \lambda_n^{(n)} = \varrho_n^{(n)}.$$

We say that the series $\sum_{k=0}^{\infty} u_k$ is (K, φ)-summable to s, in symbols

$$(K, \varphi) \sum_{k=0}^{\infty} u_k = s,$$

if and only if

$$\lim_{n\to\infty} \sum_{k=0}^{n} \varrho_k^{(n)} u_k = \lim_{n\to\infty} \sum_{k=0}^{n} \lambda_k^{(n)} s_k = s,$$

where $s_k = \sum_{i=0}^{k} u_i$. The (K, φ)-summability is identical with Cesàro (C, 1)-summability in the case $\varphi(x) = 1$ for $x \in (0,1)$, $\varphi(0) = \varphi(1) = 0$ (see [1]).

Let $C_{2\pi}$ be the class of all continuous and 2π-periodic functions. If $f \in C_{2\pi}$ and if

$$\left| \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(x+\nu h) \right| \leqslant M|h|^{\alpha} \qquad (0 < \alpha \leqslant r), \tag{2}$$

for all x and h, $M = \text{const.}$, we write $f \in \operatorname{Lip}_M^{(r)} \alpha$.

The class of all functions $f(x)$ continuous in the interval $\langle 0,1\rangle$ and such that the inequality (2) holds for every $x, x+h \in \langle 0,1\rangle$ will be denoted by $\operatorname{Lip}_M^{(r)} \alpha$ in $\langle 0,1\rangle$.

2. First we present some results concerning the regularity (in wide sense) of (K, φ)-summability.

2.1. *Let $\varphi(x)$ be a function of bounded variation in $\langle 0,1\rangle$. Then*

$$\sum_{k=0}^{\infty} u_k = s \quad \textit{implies} \quad (K, \varphi) \sum_{k=0}^{\infty} u_k = s \quad (s \textit{ finite}).$$

This follows at once from the well-known theorem of Toeplitz ([3], pp. 10—13). Namely,

$$\lambda_k^{(n)} = \frac{1}{A_n} \left\{ \sum_{\nu=0}^{n-k+1} \varphi\left(\frac{\nu}{n+2}\right) \left[\varphi\left(\frac{\nu+k}{n+2}\right) - \varphi\left(\frac{\nu+k+1}{n+2}\right) \right] + \right.$$

$$\left. + \varphi\left(\frac{n-k+2}{n+2}\right) \varphi(1) \right\} \qquad (k < n)$$

and

$$\lambda_n^{(n)} = \frac{1}{A_n} \sum_{\nu=0}^{2} \varphi\left(\frac{\nu}{n+2}\right) \varphi\left(\frac{\nu+n}{n+2}\right).$$

Hence,

$$|\lambda_k^{(n)}| \leqslant A_n^{-1} (LV + 3L^2) \qquad (0 \leqslant k \leqslant n),$$

where

$$L = \sup |\varphi(x)|, \qquad V = \operatorname{var} \varphi(x) \quad \text{in} \quad \langle 0,1\rangle.$$

Therefore

$$\sum_{k=0}^{n} |\lambda_k^{(n)}| = O(1) \qquad (\text{as } n \to \infty),$$

and the condition

$$\lim_{n\to\infty} \varrho_k^{(n)} = 1 \qquad (k = 0, 1, 2, \ldots)$$

is evident ([1] 15, p. 207).

2.2. *If $\varphi \in \operatorname{Lip}_{M_p}^{(p)} p$ $(p = 1, 2)$ in $\langle 0,1\rangle$, then*

$$(\mathrm{C},1) \sum_{k=0}^{\infty} u_k = s \quad \textit{implies} \quad (K, \varphi) \sum_{k=0}^{\infty} u_k = s \quad (s \textit{ finite}).$$

To prove this it is sufficient to verify the condition (A_1) of Hurwitz theorem ([3], pp. 45—6). Let $\Delta^2 \varrho_k^{(n)} = \varrho_k^{(n)} - 2\varrho_{k+1}^{(n)} + \varrho_{k+2}^{(n)}$.

We write

$$\sum_{k=0}^{n} k\,|\Delta^2 \varrho_k^{(n)}| = \sum_{k=1}^{n-2} k\,|\Delta^2 \varrho_k^{(n)}| + (n-1)\,|\varrho_{n-1}^{(n)} - 2\varrho_n^{(n)}| + n\,|\varrho_n^{(n)}|\,.$$

Since

$$\Delta^2 \varrho_k^{(n)} = \frac{1}{A_n}\left\{\sum_{\nu=0}^{n-k} \varphi\left(\frac{\nu}{n+2}\right)\left[\varphi\left(\frac{\nu+k}{n+2}\right) - 2\varphi\left(\frac{\nu+k+1}{n+2}\right) + \varphi\left(\frac{\nu+k+2}{n+2}\right)\right] + \right.$$

$$\left. + \varphi(1)\left[\varphi\left(\frac{n-k+2}{n+2}\right) - \varphi\left(\frac{n-k+1}{n+2}\right)\right] + \varphi\left(\frac{n-k+1}{n+2}\right)\left[\varphi\left(\frac{n+1}{n+2}\right) - \varphi\left(\frac{n+2}{n+2}\right)\right]\right\},$$

we have

$$|\Delta^2 \varrho_k^{(n)}| \leqslant \frac{1}{A_n}\left\{\frac{LM_2}{(n+2)^2}(n-k+1) + \frac{2LM_1}{(n+2)}\right\},$$

and

$$\sum_{k=1}^{n-2} k\,|\Delta^2 \varrho_k^{(n)}| = O(1) \qquad (\text{as } n \to \infty).$$

The relations

$$n\,|\varrho_n^{(n)}| = O(1) \quad \text{and} \quad (n-1)\,|\varrho_{n-1}^{(n)} - 2\varrho_n^{(n)}| = O(1)$$

are obvious, because $\varrho_n^{(n)} = O(A_n^{-1}) = O(1/n)$, $\varrho_{n-1}^{(n)} = O(1/n)$.

Similarly, by Hurwitz theorem, the following result can be obtained (cf. [5]).

2.3. *If* $\varphi \in \mathrm{Lip}_{M_p}^{(p)}\, p$ $(p = 1, 2, \ldots, r)$ *in* $\langle 0,1\rangle$, *and*

$$\varphi(x) = O(x^{(r-2)/2}), \qquad \varphi(1-x) = O(x^{(r-2)/2}) \quad \text{as } x \to 0,$$

then

$$(C, r-1)\sum_{k=0}^{\infty} u_k = s \qquad \text{implies} \qquad (K, \varphi)\sum_{k=0}^{\infty} u_k = s \qquad (s \text{ finite}).$$

3. Now, a theorem of Tauber's type will be given.

3.1. *Let* $\varphi \in \mathrm{Lip}_M^{(1)}\, 1$ *in* $\langle 0,1\rangle$. *If*

$$(K, \varphi)\sum_{k=0}^{\infty} u_k = s \quad \text{and} \quad \sum_{k=1}^{n} k^{\beta} u_k = o(n^{\beta}) \qquad (\text{as } n \to \infty),$$

with some positive β, *then* $\sum_{k=0}^{\infty} u_k = s$.

Proof. Write

$$s_n = \sum_{k=0}^{n} u_k, \qquad \sigma_n = \sum_{k=0}^{n} \varrho_k^{(n)} u_k, \qquad \Delta_n = s_n - \sigma_n \qquad (n = 0, 1, 2, \ldots).$$

Evidently ([1] 15, p. 209),

$$\Delta_n = \frac{1}{2A_n} \sum_{k=0}^{n} \left\{ \sum_{\nu=0}^{n-k+2} \left[\varphi\left(\frac{\nu}{n+2}\right) - \varphi\left(\frac{\nu+k}{n+2}\right) \right]^2 + \right.$$

$$\left. + \sum_{\nu=0}^{k-1} \left[\varphi^2\left(\frac{\nu}{n+2}\right) + \varphi^2\left(1 - \frac{\nu}{n+2}\right) \right] \right\} u_k;$$

whence, L being as in 2.1,

$$|\Delta_n| \leqslant \frac{M^2}{2A_n (n+2)} \sum_{k=1}^{n} k^2 |u_k| + \frac{L^2}{A_n} \sum_{k=1}^{n} k |u_k|.$$

If $nu_n \to 0$, then

$$\sum_{k=1}^{n} k^\gamma |u_k| = o(n^\gamma) \quad \text{for any positive } \gamma,$$

and hence $\Delta_n \to 0$, *i.e.* $\sum_{k=0}^{\infty} u_k = s$.

In general case, we write

$$\sigma_n = u_0 + \frac{w_n}{n^\beta} \varrho_n^{(n)} + \sum_{k=1}^{n-1} \frac{w_k}{k^\beta} \lambda_k^{(n)} - \sum_{k=1}^{n-1} w_k \left[\frac{1}{(k+1)^\beta} - \frac{1}{k^\beta} \right] \varrho_{k+1}^{(n)},$$

where $w_m = \sum_{k=1}^{m} k^\beta u_k$. Passing to limit, we obtain

$$s - u_0 = \lim_{n\to\infty} \sum_{k=1}^{n-1} w_k \left[\frac{1}{k^\beta} - \frac{1}{(k+1)^\beta} \right] \varrho_{k+1}^{(n)} =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n-1} w_k \left[\frac{1}{k^\beta} - \frac{1}{(k+1)^\beta} \right] = \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} u_k - \frac{w_{n-1}}{n^\beta} \right) = \sum_{k=1}^{\infty} u_k.$$

Remark. The last theorem is also true for any function φ of bounded variation in $\langle 0,1 \rangle$.

4. In this section we investigate the behaviour of the difference $1 - \varrho_k^{(n)}$ as $n \to \infty$.

4.1. *Suppose that $\varphi(x)$ is absolutely continuous in $\langle 0,1 \rangle$ and piecewise differentiable in every interval $\langle \delta, 1-\delta \rangle$, $\delta > 0$. If, moreover, the improper Riemann integral $\int_{0+}^{1-} \varphi'^2(x)\,dx$ exists and $|\varphi(x)| + |\varphi(1-x)| = o(x^{1/2})$ as $x \to 0+$, then*

$$1 - \varrho_k^{(n)} = \frac{k^2 \int_{0+}^{1-} \varphi'^2(x)\,dx}{2 \int_0^1 \varphi^2(x)\,dx} \cdot \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) \qquad (k = 1, 2, \ldots). \tag{3}$$

Proof. Since

$$A_n - A_k^{(n)} = \frac{1}{2} \sum_{\nu=0}^{n-k+2} \left[\varphi\left(\frac{\nu}{n+2}\right) - \varphi\left(\frac{\nu+k}{n+2}\right) \right]^2 + o\left(\frac{1}{n}\right),$$

we have

$$(n+2)(A_n - A_k^{(n)}) = \frac{k^2}{2} \int_{0+}^{1-} \varphi'^2(x)\, dx + o(1)$$

and, together with (1), the formula (3) follows (cf. [1] 15, p. 211).

4.2. Let

$$\varphi(x) = x^\alpha \quad (0 < \alpha \leqslant 1) \quad \text{for} \quad x \in \langle 0, 1/2\rangle,$$

$$\varphi(x) = \varphi(1-x) \quad \text{for} \quad x \in (1/2, 1\rangle.$$

This function satisfies all assumptions of 4.1, if $\alpha > 1/2$; therefore

$$1 - \varrho_k^{(n)} = \frac{2k^2\alpha^2(2\alpha+1)}{(2\alpha-1)n^2} + o\left(\frac{1}{n^2}\right) \quad \text{when} \quad \alpha > \frac{1}{2}.$$

Moreover, the asymptotic formulas

$$1 - \varrho_k^{(n)} = \frac{k^2 \ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right) \quad \text{when} \quad \alpha = \frac{1}{2},$$

and

$$1 - \varrho_1^{(n)} = \frac{4^\alpha(2\alpha+1)\gamma}{n^{2\alpha+1}} + o\left(\frac{1}{n^{2\alpha+1}}\right) \quad \text{when} \quad \alpha < \frac{1}{2},$$

where

$$\alpha^2[\zeta(2-2\alpha) - 1] + 1 < \gamma < \alpha^2\zeta(2-2\alpha) + 1 \quad \left(\zeta(z) = \sum_{n=1}^{\infty} 1/n^z\right),$$

may be also directly proved.

4.3. *Let*

$$|\varphi(x)| \geqslant Mx^\alpha \quad \text{or} \quad |\varphi(1-x)| \geqslant Mx^\alpha \quad \text{for} \quad x \in (0, \delta\rangle,$$

where $M > 0$, $0 < \alpha < 1/2$, $0 < \delta \leqslant 1/2$. *Then*

$$1 - \varrho_k^{(n)} \geqslant C_1/n^{2\alpha+1} \quad (k, n = 1, 2, \ldots),$$

with some positive constant C_1. *If* $\varphi \in \mathrm{Lip}_N^{(1)}\beta$ $(0 < \beta \leqslant 1)$ *is of bounded variation in* $\langle 0,1\rangle$, *and* $\varphi(0) = \varphi(1) = 0$, *then*

$$1 - \varrho_k^{(n)} \leqslant C_2/n^{\beta+1} \quad (k, n = 1, 2, \ldots),$$

with some positive constant C_2 *depending on* k.

The proof of these inequalities is immediate ([1] 15, p. 209).

5. Finally we consider the convergence of linear positive operators

$$L_n(x) = L_n(x; f) = \frac{a_0}{2} + \sum_{k=1}^{n} \varrho_k^{(n)}(a_k \cos kx + b_k \sin kx),$$

where a_k and b_k are Fourier coefficients of $f \in C_{2\pi}$. As is known ([1] 15),

$$L_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\psi_n(t-x)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x+t)+f(x-t)]\,\psi_n(t)\,dt,$$

with the positive kernel

$$\psi_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \varrho_k^{(n)} \cos kt.$$

We set

$$R_n(\alpha) = \frac{1}{\pi}\int_{-\pi}^{\pi}[g_\alpha(t) - f_\alpha(t)]\,\psi_n(t)\,dt,$$

where

$$g_\alpha(t) = \frac{1}{2}|t|^\alpha, \quad f_\alpha(t) = \frac{1}{2}\left|2\sin\frac{t}{2}\right|^\alpha \qquad (0 < \alpha \leqslant 2),$$

and

$$S_n(\alpha) = \sup_{f \in \mathrm{Lip}_1^{(2)}\alpha} \{\max_{-\pi \leqslant t \leqslant \pi} |L_n(t) - f(t)|\}.$$

5.1. *For every $\alpha \in (0,2\rangle$, there is a sequence $\{\vartheta_n\}$, $0 \leqslant \vartheta_n \leqslant 1$, such that*

$$S_n(\alpha) = L_n(0; f_\alpha) + \vartheta_n R_n(\alpha), \tag{4}$$

and

$$\frac{1-\varrho_1^{(n)}}{2^{2-\alpha}} \leqslant L_n(0; f_\alpha) \leqslant \frac{1}{2}[2(1-\varrho_1^{(n)})]^{\alpha/2}, \tag{5}$$

$$R_n(\alpha) = \begin{cases} O[(3-4\varrho_1^{(n)}+\varrho_2^{(n)})^{\alpha/2}] & \text{if} \quad \alpha \leqslant 2, \\ O[(10-15\varrho_1^{(n)}+6\varrho_2^{(n)}-\varrho_3^{(n)})^{\alpha/2}] & \text{if} \quad \alpha \leqslant 1. \end{cases} \tag{6}$$

Proof. The decomposition (4) holds, since $f_\alpha \in \mathrm{Lip}_1^{(2)}\alpha$ $(0 < \alpha \leqslant 2)$. The left-hand side of (5) is obvious:

$$L_n(0; f_\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|2\sin\frac{t}{2}\right|^\alpha \psi_n(t)\,dt \geqslant \frac{2^{\alpha-2}}{\pi}\int_{-\pi}^{\pi} 2\sin^2\frac{t}{2}\,\psi_n(t)\,dt =$$

$$= \frac{2^{\alpha-2}}{\pi}\int_{-\pi}^{\pi}(1-\cos t)\,\psi_n(t)\,dt = \frac{1-\varrho_1^{(n)}}{2^{2-\alpha}}.$$

To obtain the right-hand side of (5) we write

$$L_n(0; f_\alpha) = \frac{2^{\alpha-1}}{\pi}\int_{-\pi}^{\pi}\left\{\left|\sin\frac{t}{2}\right|^\alpha [(\psi_n(t)]^{\alpha/2}\right\}[\psi_n(t)]^{1-\alpha/2}\,dt.$$

By Hölder's inequality

$$L_n(0;f_\alpha) \leqslant \frac{2^{\alpha-1}}{\pi}\left[\int_{-\pi}^{\pi}\psi_n(t)\sin^2\frac{t}{2}\right]^{\alpha/2}\left[\int_{-\pi}^{\pi}\psi_n(t)\,dt\right]^{1-\alpha/2} =$$

$$= \frac{2^{\alpha-1}}{\pi}\left[\frac{\pi}{2}(1-\varrho_1^{(n)})\right]^{\alpha/2}\pi^{1-\alpha/2} = \frac{1}{2}[2(1-\varrho_1^{(n)})]^{\alpha/2}.$$

Now, we shall prove the relation (6). Since

$$x^\alpha - \sin^\alpha x \leqslant K(\alpha)\sin^{2+\alpha}x \qquad (0 \leqslant x \leqslant \pi/2, \quad 0 < \alpha \leqslant 2),$$

where $K(\alpha)$ denotes a positive constant, we have

$$R_n(\alpha) \leqslant \frac{2^{\alpha-1}K(\alpha)}{\pi}\int_{-\pi}^{\pi}\left|\sin\frac{t}{2}\right|^{2+\alpha}\psi_n(t)\,dt.$$

By Hölder's inequality

$$R_n(\alpha) \leqslant \frac{2^{\alpha-1}K(\alpha)}{\pi^{\alpha/2}}\left[\int_{-\pi}^{\pi}\left|\sin\frac{t}{2}\right|^{2+4/\alpha}\psi_n(t)\,dt\right]^{\alpha/2}.$$

In the case $\alpha \leqslant 2$,

$$R_n(\alpha) \leqslant \frac{1}{2}\pi^{-\alpha/2}K(\alpha)\left[\int_{-\pi}^{\pi}(1-\cos t)^2\psi_n(t)\,dt\right]^{\alpha/2}.$$

Using

$$2(1-\cos t)^2 = 3 - 4\cos t + \cos 2t,$$

we get the first relation (6).

In the case $\alpha \leqslant 1$,

$$R_n(\alpha) \leqslant \frac{1}{2}(2\pi)^{-\alpha/2}K(\alpha)\left[\int_{-\pi}^{\pi}(1-\cos t)^3\,\psi_n(t)\,dt\right]^{\alpha/2},$$

and by the formula

$$4(1-\cos t)^3 = 10 - 15\cos t + 6\cos 2t - \cos 3t,$$

the second relation (6) follows.

Remark. As it is easy to check,

$$3 - 4\varrho_1^{(n)} + \varrho_2^{(n)} = (1-\varrho_1^{(n)})\,[4 - (1-\varrho_2^{(n)})/(1-\varrho_1^{(n)})],$$

$$10 - 15\varrho_1^{(n)} + 6\varrho_2^{(n)} - \varrho_3^{(n)} = (1-\varrho_1^{(n)})\,\{6\,[4 - (1-\varrho_2^{(n)})/(1-\varrho_1^{(n)})] - [9 - (1-\varrho_3^{(n)})/(1-\varrho_1^{(n)})]\}.$$

5.2. For example, if $\varphi(x)$ satisfies the assumption of 4.1,

$$S_n(2) = \frac{\int_{0+}^{1-}\varphi'^2(x)\,dx}{2\int_0^1\varphi^2(x)\,dx}\cdot\frac{1}{n^2} + o\left(\frac{1}{n^2}\right).$$

In the case $\varphi(x) = \sin \pi x$, we can obtain a slightly improved Korovkin's result ([1] 13, pp. 101—2)

$$S_n(2) = \frac{\pi^2}{2n^2} + O\left(\frac{1}{n^3}\right).$$

It may be also observed that, for the last $\varphi(x)$, the following refinement of Petrov's result holds ([4], pp. 128—131)

$$S_n(1) = \left(\int_0^{\pi} \frac{\sin x}{x}\, dx - \frac{2}{\pi}\right)\frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Jackson's theorem yields the inequality ([2], pp. 75—6)

$$S_n(\alpha) < 3/n^{\alpha} \qquad (0 < \alpha \leqslant 1, \quad n = 1, 2, \ldots) \quad \text{for} \quad \varphi(x) = \sin \pi x.$$

We remark that direct calculation gives $S_n(\alpha) < 5^{\alpha}/[2(n+2)^{\alpha}]$ $(0 < \alpha \leqslant 2, \ n = 0, 1, 2, \ldots)$. Indeed,

$$S_n(\alpha) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |t|^{\alpha}\, \psi_n(t)\, dt \leqslant \frac{\pi^{\alpha-1}}{2}\int_{-\pi}^{\pi} \left|\sin\frac{t}{2}\right|^{\alpha} \psi_n(t)\, dt,$$

and by Hölder's inequality

$$S_n(\alpha) \leqslant \frac{\pi^{\alpha}}{2^{1+\alpha/2}}\left(1 - \varrho_1^{(n)}\right)^{\alpha/2} = \frac{\pi^{\alpha}}{2}\sin^{\alpha}\frac{\pi}{2n+4} < \frac{5^{\alpha}}{2(n+2)^{\alpha}}.$$

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MATHÉMATIQUE

Sur un problème d'interpolation pour l'équation différentielle ordinaire d'ordre n

par

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Présenté par T. WAŻEWSKI le 14 juillet 1961

1. Pour l'équation différentielle d'ordre n

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \tag{1}$$

on pose le problème d'interpolation suivant: étant donnés n points $(t_1, c_1), \ldots, (t_n, c_n)$ $(t_1 < t_2 < \ldots < t_n)$ trouver une solution $x(t)$ de l'équation (1) telle que l'on ait

$$x(t_i) = c_i \quad (i = 1, 2, \ldots, n). \tag{2}$$

Dans le cas où la fonction $f(t, x_0, \ldots, x_{n-1})$ satisfait à la condition de Lipschitz

$$|f(t, x_0, \ldots, x_{n-1}) - f(t, \bar{x}_0, \ldots, \bar{x}_{n-1})| \leqslant \sum_{i=0}^{n-1} L_i |x_i - \bar{x}_i|$$

on connait bien [3] le théorème suivant de l'unicité:

Pour que le problème (2) admette une solution au plus, il suffit que toute fonction $x(t)$ satisfaisant à l'inégalité différentielle

$$|x^{(n)}(t)| \leqslant \sum_{i=0}^{n-1} L_i |x^{(i)}(t)| \tag{3}$$

et telle que $x(t_i) = 0 \quad (i = 1, \ldots, n)$ soit identiquement nulle.

2. Dans la présente note nous nous proposons de démontrer que la même condition est suffisante pour l'existence d'une solution du problème (2) au moins, même dans un cas beaucoup plus général. Nous allons démontrer notamment le théorème suivant.

THÉORÈME. *Admettons que la fonction* $f(t, x_0, \ldots, x_{n-1})$ *soit continue dans l'ensemble*: $a < t < b$, $-\infty < x_i < +\infty \quad (i = 0, 1, \ldots, n-1)$ *et qu'elle satisfasse à l'inégalité*

$$|f(t, x_0, \ldots, x_{n-1})| \leqslant M + \sum_{i=0}^{n-1} L_i |x_i| \quad (M \geqslant 0, L_i > 0). \tag{4}$$

Si, quels que soient les nombres $t_1 < t_2 < \ldots < t_n$ *appartenant à l'intervalle* (a, b), *la fonction* $x(t) \equiv 0$ *est l'unique solution de l'inégalité* (3) *satisfaisant à* $x(t_i) = 0$ $(i = 1, \ldots, n)$, *il existe au moins une solution du problème* (2).

3. Démonstration dans le cas linéaire. On a dans ce cas

$$x^{(n)} = \sum_{i=0}^{n-1} p_i(t)\, x^{(i)} + q(t), \tag{5}$$

où les fonctions $p_i(t)$ et $q(t)$ sont continues et

$$|p_i(t)| \leqslant L_i \quad (a < t < b). \tag{6}$$

La solution générale de l'équation (5) revêt la forme

$$x(t) = \sum_{i=1}^{n} C_i u_i(t) + w(t), \tag{7}$$

$(u_1(t), \ldots, u_n(t))$ étant le système fondamental de solutions. Il est facile de voir que pour l'équation (5) le problème (2) ne peut admettre qu'une seule solution. En effet, la différence entre deux solutions s'annule aux points $t_1, \ldots, t_n$ et, en vertu de (6), satisfait à l'inégalité (3). Par conséquent, elle s'annule identiquement dans tout l'intervalle (a, b).

Il en résulte, d'après (7), que le système d'équations en $C_1, \ldots, C_n$

$$c_k = \sum_{i=1}^{n} C_i u_i(t_k) + w(t_k) \quad (k = 1, \ldots, n) \tag{8}$$

admet, quels que soient $c_1, \ldots, c_n$, une solution au plus. Donc, le déterminant $|u_i(t_k)|$ est différent de zéro, ce qui signifie que le système (8) admet toujours une solution.

Le théorème se trouve démontré pour l'équation (5).

4. Démonstration dans le cas général. En ajoutant, s'il y a lieu, aux intégrales de l'équation (1) un polynôme convenablement choisi de degré $n - 1$ au plus, on peut ramener le problème (2) au problème

$$x(t_i) = 0 \quad (i = 1, \ldots, n). \tag{9}$$

Soient c et d deux nombres tels que $a < c < t_1 < \ldots < t_n < d < b$. Il suffit de démontrer l'existence d'une solution $x(t)$ du problème (9) dans l'intervalle $\langle c, d\rangle$ car, en raison de (4), cette solution est prolongeable dans tout l'intervalle (a, b).

La fonction $f(t, x_0, \ldots, x_{n-1})$ peut être écrite comme suit:

$$f(t, x_0, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} p_i(t, x_0, \ldots, x_{n-1})\, x_i + q(t, x_0, \ldots, x_{n-1}) \tag{10}$$

où

$$p_i(t, x_0, \ldots, x_{n-1}) = f(t, x_0, \ldots, x_{n-1}) \left(M + \sum_{i=0}^{n-1} L_i |x_i|\right)^{-1} L_i\, \varepsilon(x_i),$$

$$q(t, x_0, \ldots, x_{n-1}) =$$
$$= f(t, x_0, \ldots, x_{n-1}) \Big(M + \sum_{i=0}^{n-1} L_i |x_i|\Big)^{-1} \Big(M + \sum_{i=0}^{n-1} (\eta(x_i) - \varepsilon(x_i)) L_i x_i\Big).$$

La fonction $\varepsilon(x)$ est égale à x dans l'intervalle $(-1, +1)$, à $+1$ pour $x \geqslant 1$ et à -1 pour $x \leqslant -1$. La fonction $\eta(x)$ est égale à 1 pour $x \geqslant 0$ et à -1 pour $x < 0$.

Les fonctions p_i et q sont continues dans l'intervalle (a, b) et, en vertu de (4), satisfont aux inégalités

$$(11) \qquad |q(t, x_0, \ldots, x_{n-1})| \leqslant M + \sum_{i=0}^{n-1} L_i, \qquad |p_i(t, x_0, \ldots, x_{n-1})| \leqslant L_i.$$

Nous allons envisager la transformation T dans l'espace fonctionnel E de fonctions $x(t)$ de classe C^{n-1} dans l'intervalle $\langle c, d\rangle$, muni de la norme habituelle

$$\|x(t)\| = \sup_{\langle c, d\rangle} \sum_{i=0}^{n-1} |x^{(i)}(t)|,$$

La transformation T est définie de façon suivante: à toute fonction $x(t) \in E$ on fait correspondre la solution $y(t) \in E$ de l'équation linéaire

$$(12) \qquad y^{(n)} = \sum_{i=0}^{n-1} p_i(t, x(t), \ldots, x^{(n-1)}(t))\, y^{(i)} + q(t, x(t), \ldots, x^{(n-1)}(t))$$

satisfaisant aux conditions $y(t_i) = 0$ $(i = 1, \ldots, n)$. En raison des inégalités (11) ces conditions déterminent la fonction $y(t)$ de façon univoque. En tenant compte de la formule (10) on voit aisément que pour démontrer notre théorème il suffit de prouver que la transformation T admet un point invariant au moins. Or, l'existence d'un tel point résulte du théorème bien connu de Schauder [2] car, comme nous allons démontrer, l'ensemble $T(E)$ est compact dans E et la transformation T est continue.

5. Les fonctions $y(t) \in T(E)$ satisfont dans l'intervalle $\langle c, d\rangle$ à l'inégalité différentielle

$$(13) \qquad |y^{(n)}(t)| \leqslant R + \sum_{i=0}^{n-1} L_i |y^{(i)}(t)| \qquad \Big(R = M + \sum_{i=0}^{n-1} L_i\Big)$$

et aux conditions

$$(14) \qquad y(t_1) = y(t_2) = \ldots = y(t_n) = 0.$$

Nous allons démontrer que l'ensemble F de toutes les fonctions de classe C^n satisfaisant aux conditions (13) et (14) est compact dans E. Deux cas sont à distinguer: ou bien il existe une constante N telle que pour toute fonction $y(t) \in F$ il existe un $\tau \in \langle c, d\rangle$ tel que

$$(15) \qquad \sum_{i=0}^{n-1} |y^{(i)}(\tau)| \leqslant N,$$

ou bien, quel que soit N, il existe une fonction $y_N(t) \in F$ telle que

$$\sum_{i=0}^{n-1} |y_N^{(i)}(t)| \geqslant N \tag{16}$$

dans tout l'intervalle $\langle c, d\rangle$. Dans le premier cas on tire de (13) l'inégalité

$$\sum_{i=0}^{n-1} |y^{(i+1)}(t)| \leqslant R+(L+1)\sum_{i=0}^{n-1} |y^{(i)}(t)| \tag{17}$$

avec $L = \max L_i$. De la théorie des inégalités différentielles ([1], p. 151) et des inégalités (15), (17) il résulte que dans tout l'intervalle $\langle c, d\rangle$ on a

$$\sum_{i=0}^{n-1} |y^{(i)}(t)| \leqslant (N+R)\exp(L+1)(d-c)$$

et enfin, en vertu de l'inégalité (13),

$$|y^{(n)}(t)| \leqslant R+L(N+R)\exp(L+1)(d-c).$$

Il en résulte immédiatement que l'ensemble F est compact dans E. A plus forte raison il en est de même de l'ensemble $T(E) \subset F$.

Dans le second cas on tire des inégalités (13) et (16)

$$|y_N^{(n)}(t)| \leqslant R+\sum_{i=0}^{n-1} L_i |y_N^{(i)}(t)| \leqslant \left(1+\frac{R}{N}\right)\sum_{i=0}^{n-1} L_i |y_N^{(i)}(t)|.$$

En posant $h = (b-a)/(d-c)$ et

$$s = a+h(t-c), \qquad z(s) = y_N(c+h^{-1}(s-a))$$

on obtient l'inégalité

$$h^n |z^{(n)}(s)| \leqslant \sum_{i=0}^{n-1} L_i \left(1+\frac{R}{N}\right) h^i |z^{(i)}(s)| \qquad (a < s < b),$$

d'où, en supposant N suffisamment grand, on tire

$$|z^{(n)}(s)| \leqslant \sum_{i=0}^{n-1} L_i |z^i(s)| \qquad (a < s < b).$$

D'autre part, en posant $s_i = a+h(t_i-c) \quad (i = 1, \ldots, n)$ on obtient une suite $s_1 < s_2 < \ldots < s_n$ de nombres de l'intervalle (a, b) tels que $z(s_1) = z(s_2) = \ldots = z(s_n) = 0$. Il en vient que la fonction $z(s)$ est identiquement nulle dans l'intervalle (a, b) et, par conséquent, il en est de même de la fonction $y_N(t)$ dans l'intervalle (c, d) ce qui est en contradiction avec l'inégalité (16). Le second cas est donc impossible ce qui achève la démonstration que l'ensemble $T(E)$ est compact dans l'espace E.

6. Il nous reste à démontrer que la transformation T est continue.

Soit $\{x_k(t)\}$ une suite de fonctions de classe C^{n-1} telles que l'on ait

$$\lim_{k\to\infty} x_k^{(i)}(t) = x_0^{(i)}(t) \qquad (i = 0, \ldots, n-1)$$

uniformément dans l'intervalle $\langle c, d\rangle$. Posons $y_k(t) = Tx_k(t) \quad (k = 0, 1, 2, \ldots)$. On a donc

$$y_k^{(n)}(t) = \sum_{i=0}^{n-1} p_i(t, x_k(t), \ldots, x_k^{(n-1)}(t))\, y_k^{(i)}(t) + \\ + q(t, x_k(t), \ldots, x_k^{(n-1)}(t)) \qquad (k = 0, 1, \ldots),$$

et, par conséquent,

$$y_k^{(n)}(t) - y_0^{(n)}(t) = \sum_{i=0}^{n-1} p_i(t, x_k(t), \ldots)\,(y_k^{(i)}(t) - y_0^{(i)}(t)) + \varepsilon_k(t)$$

où

$$\varepsilon_k(t) = q(t, x_k(t), \ldots) - q(t, x_0(t), \ldots) + \\ + \sum_{i=0}^{n-1} (p_i(t, x_k(t), \ldots) - p_i(t, x_0(t), \ldots)\, y_0^{(i)}(t).$$

Les fonctions $u_k(t) = y_k(t) - y_0(t)$ satisfont aux conditions (14) et à l'inégalité différentielle

$$|u_k^{(n)}(t)| \leqslant \sum_{i=0}^{n-1} L_i |u_k^{(i)}(t)| + \varepsilon_k$$

où $\varepsilon_k = \max_{\langle c, d\rangle} |\varepsilon_k(t)|$. Il vient de la dernière inégalité que les fonctions $u_k(t)/\varepsilon_k$ satisfont à l'inégalité (13) avec $R = 1$. Cela signifie, comme nous l'avons démontré au paragraphe précédent, que ces fonctions, ainsi que leurs dérivées jusqu'à l'ordre $n - 1$, sont bornées dans leur ensemble dans tout l'intervalle $\langle c, d\rangle$. Comme ε_k tend vers zéro lorsque k croit indéfiniment, il en vient

$$\lim_{k\to\infty} y_k^{(i)}(t) = y_0^{(i)}(t) \qquad (i = 0, \ldots, n-1)$$

uniformément dans l'intervalle $\langle c, d\rangle$. Donc, la transformation T est continue et le théorème se trouve ainsi démontré.

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MATHÉMATIQUE

Sur une équation différentielle presque-périodique sans solution presque-périodique

par

Z. OPIAL

Présenté par T. WAŻEWSKI le 14 juillet, 1961

1. Dans la théorie des équations différentielles périodiques on démontre [5] que si l'équation

$$\frac{dy}{dx} = g(x, y),$$

(la fonction $g(x, y)$ étant supposée continue et périodique en x), admet une solution bornée dans l'intervalle $(-\infty, +\infty)$, alors une solution au moins de cette équation est périodique. On pourrait penser qu'un théorème analogue est vrai pour des équations et des solutions presque-périodiques. Dans la présente note nous nous proposons de donner un exemple montrant qu'il n'en est rien. Nous allons construire une équation presque-périodique dont toutes les solutions sont bornées dans l'intervalle $(-\infty, +\infty)$, mais dont aucune n'est presque-périodique.

2. Envisageons l'équation différentielle

$$\frac{d\theta}{d\varphi} = A(\varphi, \theta) \tag{1}$$

et supposons la fonction $A(\varphi, \theta)$ définie et continue dans tout le plan P des (φ, θ) et doublement périodique de période 1, c'est-à-dire telle que $A(\varphi+p, \theta+q) = A(\varphi, \theta)$, quels que soient φ, θ et les entiers p, q. Admettons, en plus, que par tout point (φ_0, θ_0) de P il passe une seule solution $\theta = f(\varphi; \varphi_0, \theta_0)$ de l'équation (1). Pour abréger les notations nous écrirons $f(\varphi; \theta_0)$ au lieu de $f(\varphi; 0, \theta_0)$.

La fonction $A(\varphi, \theta)$ étant doublement périodique, l'équation (1) peut être considérée comme définie sur le tore circulaire S, où φ désigne la longitude et θ la latitude. Cela étant, supposons que les trajectoires de cette équation réalisent le cas singulier de Poincaré, c'est-à-dire admettons qu'elles sont non-denses sur S et qu'aucune d'elles n'est fermée.

L'ensemble R d'accumulation de la suite $\{f(n; \theta_0)+m\}$ $(n, m = 0, \pm 1, \pm 2, \ldots)$ ne dépend pas de θ_0 et, dans notre hypothèse, il est parfait et totalement discon-

tinu. Pour tout θ_0 et tout $\theta_1 \in R$ il existe donc deux suites d'entiers $\{p_n\}$, $\{q_n\}$, $(n = 1, 2, \ldots)$ telles que

$$\lim_{n\to\infty} [f(p_n; \theta_0)+q_n] = \theta_1. \tag{2}$$

Par conséquent, en vertu de la dépendance continue des solutions de l'équation (1) des valeurs initiales, on a

$$\lim_{n\to\infty} [f(\varphi+p_n; \theta_0)+q_n] = f(\varphi; \theta_1), \tag{3}$$

la convergence étant uniforme dans tout intervalle borné.

Notons enfin que pour les extrémités θ_1, θ_2 d'un intervalle contigu à l'ensemble R on a forcément

$$\lim_{\varphi\to\infty} [f(\varphi; \theta_2) - f(\varphi; \theta_1)] = 0. \tag{4}$$

3. Désignons par α le coefficient de rotation de l'équation (1). Dans nos hypothèses α est nécessairement irrationnel. On démontre (cf. p. ex. [2]) que, quel que soit θ_0, la fonction

$$w(\varphi; \theta_0) = f(\varphi; \theta_0) - \alpha\varphi \tag{5}$$

est bornée dans tout l'intervalle $(-\infty, +\infty)$.

Introduisons la nouvelle fonction inconnue $w = \theta - \alpha\varphi$ pour laquelle on obtient de l'équation (1) l'équation différentielle

$$\frac{dw}{d\varphi} = B(\varphi, w), \tag{6}$$

avec $B(\varphi, w) = A(\varphi, w+\alpha\varphi) - \alpha$. A toute solution $f(\varphi; \theta_0)$ de l'équation (1) la formule (5) fait correspondre une solution $w(\varphi; \theta_0)$ de l'équation (6) et réciproquement. De la remarque faite ci-dessus il résulte que toutes les solutions de l'équation (6) sont bornées dans tout l'intervalle $(-\infty, +\infty)$.

4. Il est facile de voir que la fonction $B(\varphi, w)$ ou, ce qui revient au même, la fonction $A(\varphi, w+\alpha\varphi)$ est presque-périodique en φ, uniformément par rapport à w variant de $-\infty$ à $+\infty$. En effet, la fonction $A(\varphi, \theta)$ étant uniformément continue dans tout le plan P, à tout $\varepsilon > 0$ on peut faire correspondre un $\delta > 0$ tel que l'inégalité $|\theta_1 - \theta_2| < \delta$ entraîne l'inégalité

$$|A(\varphi, \theta_1) - A(\varphi, \theta_2)| < \varepsilon, \tag{7}$$

quels que soient θ_1, θ_2 et φ. D'autre part ([1], p. 54), à tout $\delta > 0$ on peut faire correspondre un $l(\delta) > 0$ tel que tout intervalle de longueur $l(\delta)$ contienne au moins un entier p tel que, pour un autre entier q convenablement choisi, on ait $|\alpha p - q| < \delta$. Donc, en appliquant l'inégalité (7) à $\theta_1 = w+\alpha\varphi$ et $\theta_2 = w+\alpha\varphi + (\alpha p - q)$, on obtient

$$|A(\varphi+p, w+\alpha(\varphi+p)) - A(\varphi, w+\alpha\varphi)| =$$
$$= |A(\varphi, w+\alpha\varphi+(\alpha p - q)) - A(\varphi, w+\alpha\varphi)| < \varepsilon,$$

quels que soient φ et w. Cela signifie que $A(\varphi, w+\alpha\varphi)$ est presque-périodique en φ, uniformément par rapport à w.

5. Il reste à démontrer qu'aucune des solutions de l'équation (6) n'est presque-périodique. Pour la démonstration par l'absurde, supposons que, pour un θ_0, la fonction $w(\varphi; \theta_0)$ soit presque-périodique. Soit (θ_1, θ_2) un des intervalles contigus à l'ensemble R. θ_1 appartient donc à l'ensemble R et, par conséquent, on peut choisir deux suites d'entiers $\{p_n\}$, $\{q_n\}$ $(n = 1, 2, \ldots)$ de telle sorte que l'on ait la relation (2). D'autre part, d'après le théorème de Bochner (v. p. ex. [4], Ch. III), la fonction $w(\varphi; \theta_0)$ étant supposée presque-périodique, de la suite $\{w(\varphi + p_n; \theta_0)\}$ on peut extraire une suite qui converge, uniformément dans tout l'intervalle $(-\infty, +\infty)$, vers une fonction également presque-périodique. Pour ne pas compliquer les notations admettons que la suite envisagée jouit elle-même de cette propriété. En vertu de (3), la suite de fonctions

$$w(\varphi+p_n; \theta_0) = f(\varphi+p_n; \theta_0) - \alpha(\varphi+p_n) =$$

$$= f(\varphi+p_n; \theta_0)+q_n - \alpha\varphi - (\alpha p_n+q_n) \qquad (n = 1, 2, \ldots)$$

tend vers la fonction $f(\varphi; \theta_1) - \alpha\varphi - \beta$ où $\beta = \lim_{n\to\infty}(\alpha p_n+q_n)$. Il en vient que la fonction $w(\varphi; \theta_1) = f(\varphi; \theta_1) - \alpha\varphi$ est presque-périodique. En remplaçant θ_1 par θ_2, on démontre de même que la fonction $w(\varphi; \theta_2)$ est presque-périodique. Donc, la différence de ces deux fonctions

$$w(\varphi; \theta_2) - w(\varphi; \theta_1) = f(\varphi; \theta_2) - f(\varphi; \theta_1)$$

est aussi presque-périodique ce qui est en contradiction avec (4) car une fonction presque-périodique non identiquement nulle ne peut pas tendre vers zéro lorsque $|\varphi|$ tend vers l'infini.

6. Remarquons que l'on pourrait établir le même résultat en utilisant la théorie générale des mouvements presque-périodiques d'un système autonome (voir [6], Ch. 5). Il suffirait à cet effet de montrer que, sur le tore S, les trajectoires de l'équation (1) passant par l'ensemble R' des points $(0, \theta)$ avec $\theta \in R$ ne sont pas stables au sens de Liapounoff par rapport à l'ensemble R'.

Notons que dans le cas où les trajectoires de l'équation (1) sont denses sur le tore S on démontre facilement [8] que toutes les solutions de l'équation (6) sont presque-périodiques.

Il est à remarquer que l'exemple que nous venons d'exposer montre que la réponse au problème 6 posé dans [9] est négative. A plus forte raison, il en est de même du problème 5.

Notons enfin que cet exemple montre que dans les théorèmes d'existence (voir p. ex. [3], [7]) des solutions presque-périodiques de l'équation (6) les conditions qu'on impose au second membre de cette équation, en général très restrictives, sont beaucoup plus essentielles qu'on ne puisse penser au premier regard.

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MATHEMATICS

On Diametral Approximative Dimension and Linear Homogeneity of F-spaces

by

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In this paper we introduce *) the notion of diametral approximative dimension (d.a.d.) of linear topological spaces, which expresses the rank of approximation of compact sets in the space by finite-dimensional linear manifolds. Some basic properties of d.a.d. are here proved. Diametral approximative dimension is an isomorphic invariant (invariant under linear homeomorphisms) analogous to the Kolmogorov's approximative dimension, definition of which is based on ε-capacity of compact sets in the space [6], [13]. In the case of nuclear spaces d.a.d. is more selective than approximative dimension (see [12], Th. 6).

Making use of the notion of d.a.d. we show that the space s of all scalar sequences **) is the unique non-normable F-space in which all infinite-dimensional subspaces are isomorphic and we give an example of an infinite dimensional F-space non-isomorphic with its subspace of deficiency one (cf. [1], p. 245). Following Bourbaki [4], we understand by F-space locally convex complete metric linear space $= B_0$-space in terminology of Mazur and Orlicz [9]. To prove the characterization of the space s quoted above we apply Theorem 1, of the second part of this note, concerning the characterization of such F-spaces which contain subspaces isomorphic to Köthe spaces. This theorem is also useful for the investigation of topological structure of linear metric spaces.

Preliminaries. Let X be a linear topological space over the real or complex field. Denote

$\mathfrak{U}(X)$ — the class of all neighbourhoods of zero in X,

$\mathfrak{B}(X)$ — the class of all bounded ***) sets in X.

*) The results of this paper were presented in authors report: Approximative dimensions of linear topological spaces and some its application, during the Conference on Functional Analysis in September 1960 in Warsaw.

**) In Köthe's monography [8] this space is denoted by ω.

***) The set A in X is bounded if for every neighbourhood U of the zero there is $t > 0$ such that $tU \supset A$.

If $h(x, y)$ is an arbitrary inner product defined on X, then the set $\{x: |h(x, x)| < 1\}$ will be called an *ellipsoid*. X is said to be a *generalized Hilbert space*, if there exists in X a basis of neighbourhoods of zero whose members are ellipsoids.

If A is an arbitrary set in X, then A° will denote the polar of A in X^*, i.e. $A^\circ = \{x^* \in X^*: x^* x < 1, \text{ for } x \in A\}$.

We recall [4] that the space X^* *conjugate* to X consists of all continuous linear functionals on X, the basis of neighbourhoods of zero in X^* is formed by all polars of bounded sets in X.

A sequence $(|.|_\alpha)$ $\alpha = 1, 2, \ldots$ of pseudonorms defined on an F-space X is said to be *fundamental*, if the sets $K_\alpha = \{x: |x|_\alpha < 1\}$ constitute a basis of neighbourhoods of zero in X and $|x|_1 \leqslant |x|_2 \leqslant \ldots$ for any x in X.

A generalized Hilbert F-space will be called a *countable-Hilbert space*.

An F-space X is called *normable*, if it is isomorphic to a Banach space.

Let $(a_{\alpha n})$, $n = 1, 2, \ldots$ be an infinite matrix of positive numbers such that

$$(1) \qquad a_{1n} \leqslant a_{2n} \leqslant \ldots \text{ for } n = 1, 2, \ldots; \sum_{n=1}^{\infty} \frac{a_{\alpha n}}{a_{\alpha+1, n}} < \infty \text{ for } \alpha = 1, 2, \ldots$$

The space of all scalar sequences $x = (\xi_n)$ such that

$$(2) \qquad |x|_\alpha = \sup a_{\alpha n} |\xi_n|,$$

with the topology given by the fundamental system of pseudonorms (2) will be called the *Köthe space* $M(a_n)$ *).

Let (a_n), (b_n) and (t_α) be scalar sequences such that $a_1 \leqslant a_2 \leqslant \ldots$; $1 > b_1 \geqslant b_2 \geqslant \ldots$; $t_1 > t_2 > \ldots$, $\lim_{\alpha \to \infty} t_\alpha = 0$. If matrices $(a_{\alpha n}) = (a_n^\alpha)$ and $(b_{\alpha n}) = (b_n^{t_\alpha})$ fulfil the condition (1), then the spaces $M(a_{\alpha n})$ and $M(b_{\alpha n})$ are said to be power spaces of infinite type and of finite type, respectively.

I. Diametral approximative dimension. Let A and B be arbitrary sets in a linear space X and L — a linear set. Denote

$$\delta(A, B, L) = \inf \{\varepsilon > 0: L + \varepsilon B \supset A\}.$$

Let **)

$$\delta_n(A, B) = \inf \delta(A, B, L),$$

the inf being taken over all n-dimensional linear subsets L. Now we define a functor

$$\delta(A, B) = \{(t_n): t_n / \delta_n(A, B) \to 0\}$$

which to every pair A, B of sets in X assigns a class of scalar sequences.

*) The class of Köthe spaces coincides with nuclear "Stufenräume" Köthe [8], § 30.8 (see Grothendieck [5], Chap. II p. 59, Prop. 8).

**) The quantity $\delta_n(A, B)$ is called n-dimensional diameter of the set A with respect to B. Diameters of various functional classes are compared in the review-paper of Tikhomirov [14] For the estimations between diameters and capacities see [11], Th. 6 and references in [14].

The following properties are obvious

(3) If $A' \subset A$, $B \subset B'$ then $\delta(A', B') \subset \delta(A, B)$.

(4) $\delta(\lambda A, \mu B) = \delta(A, B)$ for $\lambda, \mu > 0$.

DEFINITION 1. Let X be a topological linear space. The class of scalar sequences

$$\delta(X) = \bigcup_{U \in \mathfrak{U}(X)} \bigcup_{B \in \mathfrak{B}(X)} \delta(B, U)$$

will be called the *diametral approximative dimension* (d.a.d.) of space X.

PROPOSITION 1. *If X and Y are isomorphic then $\delta(X) = \delta(Y)$. If X is a generalized Hilbert space and Y is a subspace of X, then $\delta(X) \supset \delta(Y)$. If X and Y are F-spaces and Y is a continuous linear image of X, then $\delta(X) \supset \delta(Y)$.*

The proof of this proposition is not difficult and we leave it to the reader.

PROPOSITION 2. *Let*

$$\widetilde{\delta}(X) = \bigcup_{U \in \mathfrak{U}(X)} \bigcap_{V \in \mathfrak{B}(X)} \delta(V, U). \qquad \widetilde{\delta}^*(X) = \bigcup_{B \in \mathfrak{U}(X)} \bigcap_{A \in \mathfrak{B}(X)} \delta(B, A).$$

If X is an F-space, then $\delta(X) = \widetilde{\delta}(X)$, $\delta(X^*) = \widetilde{\delta}^*(X^*)$.

Proof. 1°. From the definition of bounded sets, by (3) and (4) it follows that for arbitrary linear topological space Y, $\widetilde{\delta}(Y) \subset \delta(Y)$, $\widetilde{\delta}^*(Y) \subset \delta(Y)$.

2° Inclusion $\delta(X) \subset \widetilde{\delta}(X)$. Let U_0 be a fixed neighbourhood of zero and let

(5) $$(t_n) \in \bigcap_{V \in \mathfrak{U}(X)} \delta(V, U_0).$$

Denote by (V_k) a countable basis of neighbourhoods of zero in X. By (5), $t_n/\delta_n(V_k, U_0) \to 0$, for every k. Hence, there exist indices $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_n \to \infty$, such that $\lim_{n \to \infty} t_n/\delta_n(V_{k_n}, U_0) = 0$. For every n there is a finite set Z_n such that $Z_n \subset V_{k_n}$ and $\delta_n(Z_n, U_0) \geqslant \frac{1}{2}\delta_n(V_{k_n}, U_0)$. The set $B = \bigcup_{n=1}^{\infty} Z_n$ is bounded (has the unique limit point 0) and such that $t_n/\delta_n(Z, U_0) \to 0$. We have shown that if $(t_n) \in \bigcap_{V \in \mathfrak{a}(X)} \delta(V, U_0)$ then there exists B in $\mathfrak{b}(X)$ such that $(t_n) \subset \delta(B, U_0)$. Now, since U_0 was arbitrary, we have $\delta(X) \subset \widetilde{\delta}(X)$.

3° Proof of the inclusion $\widetilde{\delta}^*(X^*) \supset \delta(X^*)$ follows from the considerations "dual" to that of 2°.

COROLLARY 1. *Let X be an F-space, (U_n) — a basis of neighbourhoods of zero in X. Then*

$$\delta(X) = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (U_m, U_n), \qquad \delta(X^*) = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (U_n^0, U_m^0).$$

COROLLARY 2. *Let X be an F-space. $\delta(X)$ does not consist of all sequences tending to zero if and only if X is a Schwartz space* *).

PROPOSITION 2. *If X is a generalized Hilbert space, then $\delta(X) = \delta(X^*)$.*

*) An F-space X is called a *Schwartz space* if every continuous linear mapping from X to a Banach space transforms a neighbourhood of X onto a conditionally compact set.

The proof follows from the fact that if A and B are arbitrary ellipsoids in X, then $\delta_n(A, B) = \delta_n(B^\circ, A^\circ)$.

LEMMA 1. *If* $A = \{(t_n): |t_n| < a_n\}$, *with* $a_1 < a_2 < \ldots$, $B = \{(t_n): \sup_n |t_n| < 1\}$, *then* $\delta_n(B, A) = a_n$ *for* $n = 1, 2, \ldots$

The proof follows from the result of Krasnosielski—Krein—Millman [7] and Tikhomirov ([14], Th. 1).

Lemma 1 and Corollary 1 supply the simple methods of computing of d.a.d. for all *Köthe spaces*. In particular, we have:

PROPOSITION 3. *If* X *is a power space of infinite type, then* $\delta(X)$ *coincides with* X. *If* $X = M(b_{an})$ *is a power space of finite type, then* $\delta(X) = \{(t_n): t_n/b_{an} \to 0\}$, *i.e.* $\delta(X)$ *coincides with the space* X^* *in its coordinate-representation.*

COROLLARY 3. *The space* $M(\exp \alpha 2^n)$ *is not isomorphic to its maximal subspace*: $M(\exp \alpha 2^{n+1})$.

In fact, the sequence $(\exp(-2^{n+1/2}))$ belongs to the first space and not to the second one. Thus, by Proposition 3, $\delta(M(\exp \alpha 2^n)) \neq \delta(M(\exp \alpha 2^{n+1}))$, and, by Proposition 1, these spaces are not isomorphic.

II. Homogeneity problem for *F*-spaces.

DEFINITION 2. Let X be a linear topological space. X will be called *linearly homogeneous* if every infinite dimensional (closed) subspace of X is isomorphic to X. X will be called *d.a.d.-homogeneous* if for every infinite dimensional subspace Y of X we have $\delta(X) = \delta(Y)$.

LEMMA 2. *Let* Y *be an F-space with a fundamental sequence of pseudonorms* $(|.|_n)$ *satisfying the condition*

$$\sup_{|x|_n = 1} |x|_{n+1} = \infty \quad (n = 1, 2, \ldots) \tag{6}$$

Then for any $\varepsilon > 0$ *there is an* x *in* Y *such that*

$$|x|_1 > 1; \quad |x|_n/|x|_{n+1} < \varepsilon, \quad \text{for} \quad n = 1, 2, \ldots \tag{7}$$

Proof. By (6) we can define by induction a sequence (y_i) such that

$$|y_1|_1 > 2; \quad |y_i|_n < 2^{-i} \text{ for } i > n \quad \text{and} \quad n = 1, 2, \ldots, \tag{8}$$

$$|y_n|_n < (1 + \varepsilon^{-1})\left(1 + \sum_{i=1}^{n-1} |y_i|_n\right). \tag{9}$$

Let us put $x = \sum_{i=1}^{\infty} y_i$. By (8) and (9), we have $|x|_1 \geqslant |y_1|_1 - \sum_{i=2}^{\infty} |y_i|_1 > 1$, whence

$$\frac{|x|_n}{|x|_{n+1}} < \frac{\sum_{i=1}^{n} |y_i|_n + \sum_{i=n+1}^{\infty} |y_i|_n}{|y_{n+1}|_{n+1} - \sum_{i=1}^{n} |y_i|_{n+1} - \sum_{i=n+2}^{\infty} |y_i|_{n+2}} \leqslant$$

$$\leqslant \frac{\sum_{i=1}^{n} |y_i|_{n+1} + 1}{(1 + \varepsilon^{-1})\left(1 + \sum_{i=1}^{n} |y_i|_{n+1}\right) - \sum_{i=1}^{n} |y_i|_{n+1} - 1} < \varepsilon \quad \text{for} \quad n = 1, 2, \ldots.$$

THEOREM 1. *Let X be a non normable F-space. Then the following conditions are equivalent:*

(i) *X is isomorphic to no Cartesian product $E\times s$ of a B-space E by the space s — of all scalar sequences.*

(ii) *There is in X a fundamental sequence of pseudonorms $(|.|_n)$ such that*

(*) *$\sup\{|x|_{n+1}: x\in Y,\ |x|_n = 1\} = \infty$, for any subspace Y of a finite deficiency in X $(n = 1, 2, \ldots)$.*

(iii) *Every fundamental sequence of pseudonorms in X contains a subsequence satisfying* (*).

(iv) *X contains a subspace isomorphic to a Köthe space.*

Proof. non (ii) $\Rightarrow$ non (i). If (ii) is not satisfied, then there exist in X a fundamental sequence of pseudonorms $(|.|_i)$ and a sequence (Y_i) of X such that

$$X = Y_0 \supset Y_1 \supset \ldots. \tag{10}$$

$$\dim X/Y_n < \infty \quad (n = 1, 2, \ldots), \tag{11}$$

$$\sup\{|x|_{n+1}: x\in Y_n,\quad |x|_n = 1\} < \infty \quad (n = 1, 2, \ldots), \tag{12}$$

$$\text{if } Y \text{ is a subspace of } X \text{ and } \dim X/Y < \dim X/Y_n, \text{ then } \sup\{|x|_{n+1}: x\in Y,\ |x|_n = 1\} = \infty. \tag{13}$$

It is easily seen that if subspaces Y_n fulfil the conditions (10)—(13), then there is a subspace R_n such that

$$Y_{n-1} = Y_n + R_n,\ Y_n\cap R_n = \{0\} \text{ and } |r|_n = 0 \text{ for any } r\in R_n,\ n = 1, 2, \ldots. \tag{14}$$

Let us put $E = \bigcap_{n=1}^{\infty} Y_n$ and $R = \overline{\bigcup_{n=1}^{\infty}(R_1+R_2+\ldots+R_n)}$ (the bar denotes the closure). Since Y_n are closed, E is closed, too. By (12) the pseudonorms $|.|_n$ and $|.|_{n+1}$ are equivalent on Y_n. Hence, the pseudonorm $|.|_n$ is equivalent to $|.|_1$ on E $(n = 1, 2, \ldots)$. Thus, E is isomorphic to a B-space. Now it is easy to check that all quotient spaces $(R, |.|_n)$, for $n = 1, 2, \ldots$, are of a finite dimension. Hence, by [2], Th. 3, R is isomorphic to s. To complete the proof we shall show that the subspaces E and R are complementary in X. Indeed, by (14), every x in X can be written uniquelly as a sum $x = y_n+r_1+r_2+\ldots+r_n$, where $r_i\in R_i$, $y_n\in Y_n$, for $i, n = 1, 2, \ldots$ Since $|r_n|_i = 0$ for $i > n$, the series $\sum_{n=1}^{\infty} r_n$ converges. Obviously, $r = \sum_{n=1}^{\infty} r_n \in R$ and $y = \lim y_n = x - \sum_{n=1}^{\infty} r_n \in E$. Hence, every x in X may be represented as $x = e+r$, with $e\in E$, $r\in R$. Since $R\cap E = \{0\}$, this representation is unique.

(ii) $\Rightarrow$ (iii). This implication readily follows from the well-known fact that if $(|.|_i)$ is a fundamental system of in a F-space X and $\|.\|$ is an arbitrary positive-homogeneous and continuous subadditive functional on X, then there is a constant C and an index i_0 such that $\|x\| \leqslant C|x|_{i_0}$ for any x in X.

(ii) $\Rightarrow$ (iv). Without loss of generality we may suppose that X is separable. Let (x_m^*) be a sequence of linear functionals in X with the property that every sequence (x_n) in X satisfying the condition

$$(**) \qquad \limsup_{n\to\infty} |x_n|_1 > 0 \quad \text{and} \quad \lim_{n\to\infty} x_m^* x_n = 0 \qquad (m = 1, 2, \ldots)$$

contains a subsequence (x_{k_n}) which is a basis for a subspace of X. (The existence of the sequence (x_n) follows from ([3], Th. 2) and the fact that every separable F-space may be isomorphically embedded in an F-space with a basis: e.g. in the space $C(-\infty, \infty)$ of all continuous real-valued functions defined on the real axis [9]. Let us put $X_n = \bigcap_{i=1}^{n} \{x(-X : x_i^* x = 0\}$ $n = 1, 2, \ldots$

Let $(|.|_\alpha)$ be a fundamental system of pseudonorms satisfying $(*)$. Since $\dim X/X_k \leqslant k < \infty$, the space $Y = X_k$ fulfils the assumption of Lemma 2. Now, by the Lemma, there is x_n in X_n such that

$$(15) \qquad |x_n|_1 > 1, \quad |x_n|_i/|x_n|_{i+1} < 2^{-n} \qquad (n, i = 1, 2, \ldots).$$

Obviously, the sequence (x_n) satisfies $(**)$. Hence there exists a subsequence (x_{k_n}) being a basis in a subspace Z of X. To complete the proof it is enough to check that the mapping $A(t_n) = \sum_{n=1}^{\infty} t_n x_{k_n}$ is an isomorphism of the space $M(|x_{k_n}|_\alpha)$ onto Z.

The implication (iii) $\Rightarrow$ (i) is trivial.

(iv) $\Rightarrow$ (ii) — follows from the fact that the system (2) of pseudonorms of a Köthe space does not satisfy the condition $(*)$.

COROLLARY 5. *Every non-normable F-space contains an infinite-dimensional nuclear subspace.*

PROPOSITION 4. *A separable F-space X is d.a.d.-homogeneous if and only if it is isomorphic to a Banach space or to the space s.*

This Proposition follows from Corollary 2, ([10], 3.1), Theorem 1 and the fact that *for every Köthe space $M(a_{an})$ there exists a subsequence (k_n) of indices such that $\delta(M(a_{an})) \neq \delta(M(a_{ak_n}))$.*

COROLLARY 6. *Every non-normable linearly homogeneous F-space is isomorphic to s.*

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MATHEMATICS

Concerning the Dimension of ANR-sets

by

K. BORSUK

Presented on July 21, 1961

The purpose of the present note is to prove the following

THEOREM. *Let $\{K_\alpha\}$ be a family of n-dimensional ANR-sets lying in an n-dimensional ANR-set X and indexed by $\alpha \in A$, where A is an uncountable set. Then there exist two different indices $\alpha, \alpha' \in A$ such that dim $(K_\alpha \cap K_{\alpha'}) = n$.*

Proof. We can assume that X is a subset of the Hilbert space H^ω. Since X is an *ANR*-set, we infer that

(1) There exists a neighbourhood U of X in H^ω and a retraction $r\colon U \to X$ of U to X.

Evidently, we can assume that for distinct indices α, α' the sets K_α and $K_{\alpha'}$ are different. Since dim $K_\alpha = n$ for every $\alpha \in A$, there exists ([1], p. 195) a positive number ε_α, a compactum $B_\alpha \subset K_\alpha$ and an infinite n-dimensional chain*) $\{\varkappa_{\alpha i}\}$ such that

(2) The infinite cycle $\{\partial \varkappa_{\alpha i}\}$ lies in B_α and it is not homologous to zero in the generalized ball $Q_\alpha = \underset{x \in K_\alpha}{E}\,[\varrho(x, B_\alpha) < \varepsilon_\alpha]$.

It follows easily that

(3) If the infinite cycle $\{\partial \varkappa_{\alpha i}\}$ is homologous to zero in a set of the form $Q_\alpha \cup C$ then dim $C \geqslant n$.

Evidently, we can replace in (2) the number ε_α by an arbitrarily given positive number $\varepsilon \leqslant \varepsilon_\alpha$. Since the set of indices A is uncountable, there exists a positive

*) We understand by an *infinite n-dimensional chain lying in a compactum B* a sequence of chains $\{\varkappa_i\}$, where $\varkappa_i = c_1 \Delta_1 + c_2 \Delta_2 + \ldots + c_{k_i} \Delta_{k_i}$ for $i = 1, 2, \ldots$, with $c_0, c_2, \ldots, c_{k_i}$ belonging to an Abelian group $\mathfrak{G}_i$ (depending in general on i) and with $\Delta_0, \Delta_2, \ldots, \Delta_{k_i}$ being oriented n-dimensional simplexes their vertices belong to B and their diameters are $< \eta_i$, where $\lim_{i=\infty} \eta_i = 0$.

If all chains $\varkappa_i$ are cycles, then $\{\varkappa_i\}$ is said to be an *infinite cycle in B*. This infinite cycle is said to be *homologous to zero in a space Y*, provided there exists an infinite $(n+1)$-dimensional chain $\{\lambda_i\}$ lying in a compactum $C \subset Y$ and such that $\partial \lambda_i = \gamma_i$ for $i = 1, 2, \ldots$

number ε such that $\varepsilon \leqslant \varepsilon_\alpha$ for α belonging to an uncountable subset A_0 of A. Replacing ε_α by ε for $\alpha \in A_0$, we get

(4) $$\varepsilon_\alpha = \varepsilon \quad \text{for every} \quad \alpha \in A_0.$$

The compacta K_α may be considered as points of the space 2^X consisting of all not empty subcompacta of X. Since 2^X is a compactum and since the set A_0 is uncountable, there exists in A_0 a sequence $\{\alpha_m\}$ of different indices such that

(5) $$\lim_{m=\infty} K_{\alpha_m} = K_\beta, \quad \text{where} \quad \beta \in A_0.$$

Moreover, we can assume that $\alpha_m \neq \beta$ for every $m = 1, 2, \ldots$ and consequently

(6) $$K_{\alpha_m} \neq K_\beta \quad \text{for every} \quad m = 1, 2, \ldots.$$

Since K_β is an ANR-set, there exists in H^ω a neighbourhood V of K_β and a retraction s of V to K_β. It follows by (5) that there exists a natural number m_0 such that for the index $\gamma = \alpha_{m_0}$ the following relations hold:

(7) The segment $\overline{xs(x)}$ is contained in $U \cap V$, for every $x \in K_\gamma$.

(8) The diameter of the set $r(\overline{xs(x)})$ is $< \varepsilon$, for every $x \in K_\gamma$.

Consider now the compactum

(9) $$M = \bigcup_{x \in B_\gamma} \overline{xs(x)}$$

and the function f_t, defined for every $x \in K_\gamma$ by the formula

(10) $$f_t(x) = r[(1-t)\,x + ts(x)] \quad \text{for every} \quad 0 \leqslant t \leqslant 1.$$

Evidently, the family of functions $\{f_t\}$ is a homotopical deformation of the set K_γ in the space X to the set K_β and it satisfies the following condition

(11) $$f_0(x) = x \quad \text{and} \quad f_1(x) = s(x) \quad \text{for every} \quad x \in K_\gamma.$$

Moreover, by (8), (9), (10), (2) and (4)

(12) $$f_t(x) \in r(M) \subset Q_\gamma \quad \text{for every} \quad x \in B_\gamma.$$

We conclude that there exists in X an infinite $(n+1)$-dimensional chain $\{\lambda_i\}$ such that

(13) $$\partial\lambda_i = \varkappa_{\gamma i} - s(\varkappa_{\gamma i}) - \mu_i,$$

where $\{\mu_i\}$ is an infinite n-dimensional chain lying in $r(M) \subset Q_\gamma$. Thus we see that $\{\varkappa_{\gamma i} - s(\varkappa_{\gamma i}) - \mu_i\}$ is an infinite n-dimensional cycle, lying in the compactum $K_\beta \cup K_\gamma \cup r(M)$ and homologous to zero in X. In order to conclude the proof, it suffices to show that the infinite cycle $\{\varkappa_{\gamma i} - s(\varkappa_{\gamma i}) - \mu_i\}$ is not homologous to zero in $K_\beta \cup K_\gamma \cup r(M)$.

Let us suppose, on the contrary, that there exists in $K_\beta \cup K_\gamma \cup r(M)$ an infinite $(n+1)$-dimensional chain $\{\vartheta_i\}$ such that

(14) $$\partial\vartheta_i = \varkappa_{\gamma i} - s(\varkappa_{\gamma i}) - \mu_i.$$

One sees easily that we can assume (applying, if it is necessary, a suitable infinitely small translation of the vertices of $\{\vartheta_i\}$) that for each simplex Δ of the chain ϑ_i all vertices of Δ belong to at least one of the sets K_γ and $K_\beta \cup r(M)$. Consequently, there exists a decomposition

$$\vartheta_i = \vartheta_i' - \vartheta_i'', \tag{15}$$

where the chain ϑ_i' lies in K_γ and the chain ϑ_i'' — in $K_\beta \cup r(M)$. We infer, by (14) and (15), that

$$\varkappa_{\gamma i} - \partial\vartheta_i' = s(\varkappa_{\gamma i}) + \mu_i - \partial\vartheta_i''.$$

The chain on the left lies in K_γ and the chain on the right — in $K_\beta \cup r(M)$. It follows that $\varkappa_{\gamma i} - \partial\vartheta_i'$ lies in the set

$$K_\gamma \cap (K_\beta \cup r(M)) = (K_\beta \cap K_\gamma) \cup (K_\gamma \cap r(M)).$$

If we observe that $\partial(\varkappa_{\gamma i} - \partial\vartheta_i') = \partial\varkappa_{\gamma i}$, we infer that the infinite cycle $\{\partial\varkappa_{\gamma i}\}$ is homologous to zero in the set $(K_\beta \cap K_\gamma) \cup (K_\gamma \cap r(M)) \subset Q_\gamma \cup (K_\beta \cup K_\gamma)$. But this contradicts (3), because $\dim(K_\beta \cap K_\gamma) < n$. Thus we have shown that the infinite cycle $\{\varkappa_{\gamma i} - s(\varkappa_{\gamma i}) - \mu_i\}$ is not homologous to zero in $K_\beta \cup K_\gamma \cup r(M)$ and our proof is concluded.

Remark 1. The Cartesian product of the Cantor discontinuum D with the n-dimensional Euclidean ball Q^n is an n-dimensional compactum containing a family of $2^{\aleph}$ disjoint n-dimensional balls of the form $(x) \times Q^n$, with $x \in D$. Consequently, the hypothesis that X is an ANR-set can not be omitted in the formulation of the theorem.

Remark 2. If the dimension of a compactum X in a point x_0 is $> n$, then there exists an $\varepsilon > 0$ such that for every positive $\alpha < \varepsilon$ the set $K_\alpha = \underset{x \in X}{E}[\varrho(x, x_0) = \alpha]$ is a compactum of dimension $\geqslant n$. Consequently, a compactum of dimension $> n$ contains always a family consisting of $2^{\aleph_0}$ disjoint compacta of dimensions $\geqslant n$.

Problem. *Is the hypothesis that the sets K_α are ANR-sets essential for the theorem?*

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Log-Periodic-Aerial-Interferometer for Radioastronomy

by

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Presented by W. IWANOWSKA on June 20, 1961

For a long time a need was felt for very wide-band simple aerials that could be used for receiver testing and radioastronomy observations. Further requirement for such aerials is absolute lack of matching-devices and adjustments. With this in mind we have constructed a log-periodic aerial interferometer [1], [2] for use at our Observatory. Our design has the following parameters: $\alpha = 60°$, $\psi = 37°$, $\tau = 0.6$ and covers frequency range from 100 to 1000 Mc/s with calculated gain about 6 db over dipole. The log-periodic structure of the aerial, made of 8 mm.

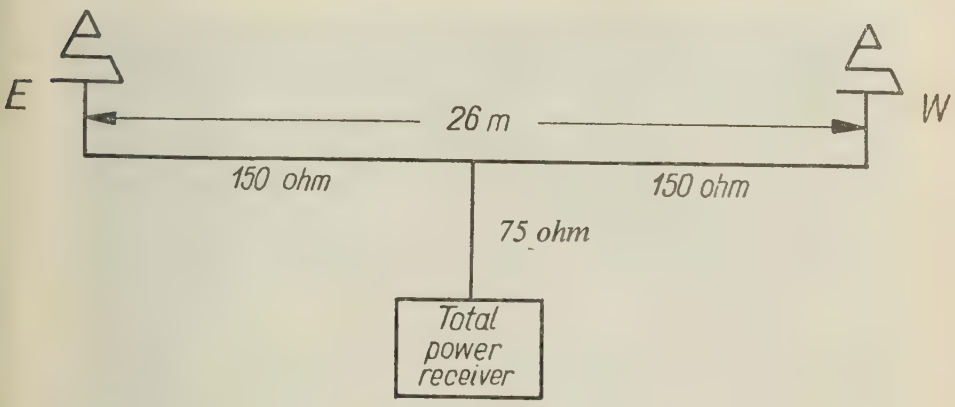

Fig. 1. The block-diagram of the log-periodic interferometer

brass-wire, was welded to the copper supporting tubes. The two aerials (Fig. 1) forming an interferometer of 26 meter spacing are connected to a wide-band junction with 150 ohm coax cable and from junction to the receiver 75 ohm coax cable was used. This insures wide-band impedance termination. The lay out of aerial can be seen from Fig. 2. Due to limited measuring facilities aerial behaviour could be checked on two frequencies only, viz. 127 Mc/s and 327 Mc/s. The mean V.S.W.R. was 1.16 at 327 Mc/s and 1.62 at 127 Mc/s. The polarization is linear as could be

found by 90° rotation around the direction of axis of maximum gain. Due to high gain the quiet Sun could be seen on both frequencies, and thanks to wide reception angle the quiet Sun could be seen for about 7 hours on both frequencies.

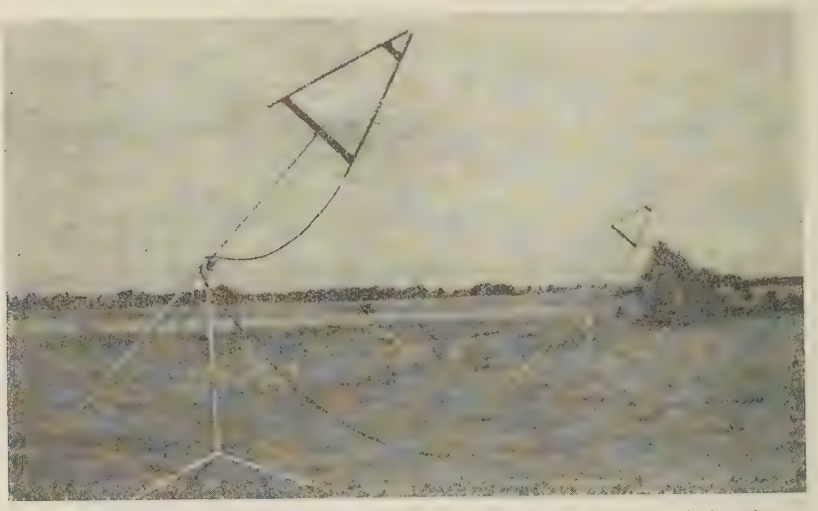

Fig. 2. The log-periodic interferometr. It can be moved in azimuth and elevation

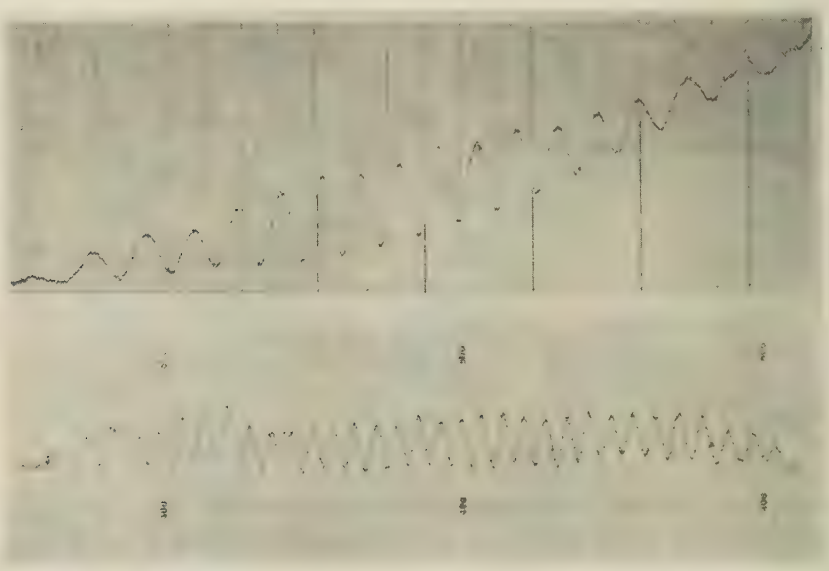

Fig. 3. The Solar interferometric records. Upper — 127 Mc/s, lower — 327 Mc/s

The Solar interference patterns can be seen from Fig. 3. Sloping of interference pattern of 127 Mc/s is due to galactic background. These records were the basis for gain measurements of the log-periodic aerials for the quiet Sun flux levels 3.1 ×

3.1×10^{-22} Watt meter^{-2} c/s^{-1} at 127 Mc/s and 8.4×10^{-22} Watt meter^{-2} c/s^{-1} at 327 Mc/s. The measured gain of a single aerial was 5.5 db for 127 Mc/s and about 8.0 db for 327 Mc/s over dipole. The 327 Mc/s gain is less certain due to the lack of information of Solar activity on this frequency.

To the advantages of the log-periodic aerials when used for radioastronomy one may count:

1) wide frequency range with ten-to-one frequency ratio,
2) constant and comparatively high gain within useful band-width.
3) wide and frequency independent reception angle about 100° giving 7 hour Solar interferometric observations without aerial movement,
4) simplicity in use, no tuning or matching of the aerial is necessary,
5) light weight and portability.

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Note on the Field Method of Obtaining the Conservation Laws and Solving the Two Body Problem in General Relativity

by

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Presented by L. INFELD on July 5, 1961

Conservation Laws analogous to those of classical physics hold to a certani approximation in general relativity. These are the laws of the conservation of energy, momentum and angular momentum (E, P^k, $J^{[ab]}$). So far [1] the method of obtaining these laws *explicitly* up to a certain order was to find the Lagrangian that gives the expressions for E, P^k and $J^{[ab]}$ through a classical-like formula.

Now let $\overset{A}{\xi}{}^a$ denote the world line of the A'th particle, the stroke denotes partial differentiation, and let x^0, x^1, x^2, x^3 be the time and space co-ordinates of a Riemannian manifold with velocity of light $c = 1$. Then we have for the Lagrangian

$$(L_{|\overset{A}{\xi}{}^a_{|0}})_{|0} - L_{|\overset{A}{\xi}{}^a} = 0,$$

where the Greek indices take values 0, 1, 2, 3, Latin — 1, 2, 3. The next step is to show, by analogy with classical physics, that

$$\sum_A \overset{A}{\xi}{}^s_{|0} (L_{|\overset{A}{\xi}{}^s_{|0}}) - L = E,$$

$$\sum_A L_{|\overset{A}{\xi}{}^k_{|0}} = P^k,$$

$$\sum_A [(L_{|\overset{A}{\xi}{}^a_{|0}}) \overset{A}{\xi}{}^b - (L_{|\overset{A}{\xi}{}^b_{|0}}) \overset{A}{\xi}{}^a] = J^{[ab]}.$$

In this paper we shall try to show that the same conservation laws, giving the same values for E, P^k, $J^{[ab]}$, follow directly from the field equations

$$\mathfrak{G}^{\mu\nu} = \sqrt{-g}\,(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R) = -8\pi \mathfrak{T}^{\mu\nu} = -8\pi \sqrt{-g}\, T^{\mu\nu},$$

where $T^{\mu\nu}$ is the energy-momentum tensor. We are to do this without additional assumptions barring only the co-ordinate conditions

$$\gamma^{0\alpha}_{|\alpha} = 0,$$

$$\gamma^{mn}_{|n} = 0,$$

where

$$\sqrt{-g}\, g^{\alpha\beta} = \eta^{\alpha\beta} + \gamma^{\alpha\beta},$$

$$(\sqrt{-g})^{-1} g_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta},$$

$$g = |g_{\alpha\beta}| = |g^{\alpha\beta}|^{-1} = |\eta^{\alpha\beta} + \gamma^{\alpha\beta}|^{-1} g^2 = |\eta^{\mu\nu} + \gamma^{\mu\nu}|.$$

We describe here the density of matter through the "good" δ functions having the following properties*)

$$\int_{-a}^{+a} \delta(x) f(x)\, dx = f(0), \qquad a > 0 \quad \text{if} \quad f(x) \text{ continuous at } 0,$$

$$\int_{-a}^{+a} \frac{\delta(x)}{|x|^p}\, dx = 0, \qquad p = 1, 2, 3, \ldots, k.$$

However, to compare the known particle concepts with the unknown calculations leading to the same concepts expressed in the field language we must quote here briefly some known results, that is the expressions for L, E, P^k and $J^{[ab]}$. Note that when we speak of any order of magnitude in our approximation we mean the order of $1/c$. It is assumed that $\xi^k_{|0}$ is of the order one and the masses begin with order two. In our reasoning it is also assumed that differentiation by x^0 raises the order of magnitude by one. Then we have

$$\underset{\to 6}{L}(\text{up to the sixth order}) = \frac{1}{2} \sum_A \overset{AA}{\mu}\, \overset{A}{\xi^s_{|0}}\, \overset{A}{\xi^s_{|0}} + \frac{1}{8} \sum_A \overset{A}{\mu}\, (\overset{A}{\xi^s_{|0}}\, \overset{A}{\xi^s_{|0}})^2 +$$

$$+ \frac{3}{4} {\sum_{A,B}}' \overset{AB}{\mu\mu}\, r^{-1}_{AB} (\overset{A}{\xi^s_{|0}}\, \overset{A}{\xi^s_{|0}} + \overset{B}{\xi^s_{|0}}\, \overset{B}{\xi^s_{|0}}) - 2 {\sum_{A,B}}' \overset{AB}{\mu\mu}\, r^{-1}_{AB}\, \overset{A}{\xi^s_{|0}}\, \overset{B}{\xi^s_{|0}} -$$

$$- \frac{1}{4} {\sum_{A,B}}' \overset{AB}{\mu\mu}\, (r_{AB|}\, \overset{B}{\xi^c}\, \overset{A}{\xi^b})\, \overset{B}{\xi^b_{|0}}\, \overset{A}{\xi^c_{|0}} + \frac{1}{2} {\sum_{A,B}}' \overset{AB}{\mu\mu}\, r^{-1}_{AB} - \frac{1}{4} {\sum_{A,B}}' \overset{AB}{\mu\mu}\, \overset{A}{(\mu} + \overset{B}{\mu})\, r^{-2}_{AB} -$$

$$- \frac{1}{6} {\sum_{A,B,C}}' \overset{ABC}{\mu\mu\mu}\, (r^{-1}_{AB}\, r^{-1}_{BC} + r^{-1}_{BC}\, r^{-1}_{CA} + r^{-1}_{CA}\, r^{-1}_{AB}),$$

We denote by r_{AB} the distance between the A and B particles and $\sum'_{AB}$ means $\sum_{\substack{AB \\ A \neq B}}$ and $\overset{A}{\mu}$ denotes the Newtonian mass of the A'th particle. We have:

$$\underset{\to 6}{E}(\text{up to the sixth order}) = \frac{1}{2} \sum_A \overset{AA}{\mu}\, \overset{A}{\xi^s_{|0}}\, \overset{A}{\xi^s_{|0}} - \frac{1}{2} {\sum_{AB}}' \overset{AB}{\mu\mu}\, r^{-1}_{AB} +$$

$$+ \frac{3}{8} \sum_A \overset{A}{\mu}\, (\overset{A}{\xi^s_{|0}}\, \overset{A}{\xi^s_{|0}})^2 + \frac{3}{4} {\sum_{AB}}' \overset{AB}{\mu\mu}\, (\overset{A}{\xi^s_{|0}}\, \overset{A}{\xi^s_{|0}} + \overset{B}{\xi^s_{|0}}\, \overset{B}{\xi^s_{|0}})\, r^{-1}_{AB} +$$

$$- 2 {\sum_{AB}}' \overset{ABA}{\mu\mu}\, \overset{B}{\xi^s_{|0}}\, r^{-1}_{AB} - \frac{1}{4} {\sum_{AB}}' \overset{AB}{\mu\mu}\, (r_{AB|}\, \overset{A}{\xi^s}\, \overset{B}{\xi^r})\, \overset{A}{\xi^s_{|0}}\, \overset{B}{\xi^r_{|0}} +$$

$$+ \frac{1}{4} {\sum_{A,B}}' \overset{AB}{\mu\mu}\, \overset{A}{(\mu} + \overset{B}{\mu})\, r^{-2}_{AB} + \frac{1}{6} {\sum_{A,B,C}}' \overset{ABC}{\mu\mu\mu}\, (r^{-1}_{AB}\, r^{-1}_{BC} + r^{-1}_{BC}\, r^{-1}_{CA} + r^{-1}_{CA}\, r^{-1}_{AB}).$$

*) Their theory is given in [1].

$$\underset{\to 5}{P^a}(\text{up to the fifth order}) = \sum_A \left(1 + \frac{1}{2}\overset{A}{\xi}{}^s_{|0}\,\overset{A}{\xi}{}^s_{|0} - \frac{1}{2}\sum_B{}' \overset{B}{\mu}\, r^{-1}_{AB}\right)\overset{A}{\mu}\,\overset{A}{\xi}{}^a_{|0} + \\ - \frac{1}{2}\sum_{A,B}{}' \overset{A}{\mu}\,\overset{B}{\mu}\,(\overset{A}{\xi}{}^a - \overset{B}{\xi}{}^a)\,(\overset{A}{\xi}{}^b - \overset{B}{\xi}{}^b)\,\overset{B}{\xi}{}^b_{|0}\, r^{-3}_{AB}.$$

Before proceeding we should like to give the definitions of what we call inertial mass $\mu_{(IN)}$

$$\mu^2_{(IN)} g_{\alpha\beta}\,\xi^\alpha_{|0}\,\xi^B_{|0} = \mu^2 = \text{const.}$$

$\overset{A}{\mu}_{(IN)}$ will denote the inertial mass of the A'th particle and

$$\overset{A}{\delta} = \delta\,(x - \overset{A}{\xi}).$$

The Einstein equation gives

$$\mathfrak{G}^{\varrho\nu} + 8\pi\,\mathfrak{T}^{\varrho\nu} = 0$$

we now break $\mathfrak{G}^{\varrho\nu}$ up into linear and non-linear expressions in $\gamma^{\alpha\beta}$

$$\mathfrak{G}^{\varrho\nu} = K^{\varrho\alpha\nu\beta}_{|\alpha\beta} + \Lambda^{\varrho\nu},$$

$$K^{\varrho\alpha\nu\beta} = \frac{1}{2}\,(\eta^{\alpha\nu}\,\gamma^{\varrho\beta} + \eta^{\varrho\beta}\,\gamma^{\alpha\nu} - \eta^{\varrho\nu}\,\gamma^{\alpha\beta} - \eta^{\alpha\beta}\,\gamma^{\varrho\nu}).$$

For point particles, whose density we shall represent by δ functions

$$\mathfrak{T}^{\varrho\nu} = \sum_A \overset{A}{\mu}_{(IN)}\,\overset{A}{\xi}{}^\varrho_{|0}\,\overset{A}{\xi}{}^\nu_{|0}\,\overset{A}{\delta}.$$

For $\varrho = 0$ the above gives, because of the coordinate condition,

$$\Lambda^{0\nu} + 8\pi\,\mathfrak{T}^{0\nu} = \Lambda^{0\nu} + 8\pi\sum_A \overset{A}{\mu}\,\overset{A}{\xi}{}^\nu_{|0}\,\overset{A}{\delta} = K^{m0\nu\beta}_{\quad|\beta m} = -\frac{1}{2}\,\gamma^{0\nu}_{\quad|mm}.$$

Now integrate over 3-dimensional space and use the divergence theorem

$$\int_\Omega \left(\Lambda^{0\nu} + 8\pi\sum_A \overset{A}{\mu}_{(IN)}\,\overset{A}{\xi}{}^\nu_{|0}\,\overset{A}{\delta}\right) dx = \int_\Sigma K^{m0\nu\beta}_{\quad|\beta}\, n_m\, ds = -\int_\Sigma \frac{1}{2}\,\gamma^{0\nu}_{\quad|m}\, n_m\, ds.$$

Here Σ is the boundary of Ω. For our purposes take Ω to be all space.

We call those three "vectors" the field, inertial and gravitational momenta, respectively

$$P^a_{(F)} + P^a_{(IN)} = P^a_{(G)}.$$

The zero components are called the field, inertial and gravitational masses, respectively

$$\mu^a_{(F)} + \mu^a_{(IN)} = \mu^a_{(G)}.$$

It is known from the proof given in [1] that if the coordinate conditions $\gamma^{0\alpha}_{\quad|\alpha} = 0$ and $\gamma^{mn}_{\quad|n} = 0$ are assumed defined as above, then $\mu_{(G)}$ is a constant. Similarly it was proved that P^k under the same coordinate conditions, and certain additional

conditions which we do not need to specify here, is always a constant (not just up to a certain approximation). However, it has not yet been shown that these are the same constants which we obtain for the Langrangian up to the sixth approximation

$$P^0 = E = \mu_{(G)}$$

$P^a = P^a$ defined through the Lagrangian.

As to the angular momentum we define it in the field theory as

$$J^{[ab]} = \frac{1}{8\pi} \int_{\Omega} (\Lambda^{0a} x^b - \Lambda^{0b} x^a)\, dx + \sum_A^A \mu (\xi^a_{|0} \overset{A}{\xi}{}^b - \xi^b_{|0} \overset{A}{\xi}{}^a)$$

and we shall also show that its value coincides up to the fifth order with that obtained from the Lagrangian. All this can be done through straightforward but somewhat prolonged calculations.

This in effect leads to a new and perhaps the most simple method of solving the problem of motion of two bodies in general relativity theory.

We know that in classical mechanics the simple method of finding the motion of two bodies consists of using the first integrals of energy and angular momentum. This corresponds in our notation to the use of E_4 and $J_3^{[ab]}$. In this way we reduce the two body problem to the one body problem. Similarly, in general relativity we can reduce the two body problem to the one body problem and obtain the perihelion motion by taking into account E_6 and $J_5^{[ab]}$.

Thus, we have shown that we can solve the problem of motion omitting the finding of the Lagrangian and the explicit writing out of the equations of motion, that is only through the conservation laws that follow from the field equations. To calculate explicitly E_6 and $J_5^{[ab]}$ (which can be shown constant for n bodies) it is necessary to know $\gamma_2^{00}, \gamma_3^{0n}, \gamma_4^{00}$, but not γ_4^{mn}.

This is a note on a paper which is to appear in full elsewhere. The authors would like to thank Professor L. Infeld for his extremely valuable help.

Summary

In this paper we derive the conservation laws up to the post Newtonian approximation straight from the field equations. Thus, we obtain a simple method of solving the equations of motion. for the body problem without the use of the Lagrangian and without writing them out explicitly.

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THEORETICAL PHYSICS

On Transitions Involving Crossing of Potential Energy Surfaces

by

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Presented by J. WEYSSENHOFF on July 18, 1961

1. Consider a system of the nuclei and electrons representing reacting quasi-molecule or stable state of polyatomic molecule. (For simplicity we shall call molecule the reacting as well as the stable system). Denote by q the electronic coordinates and by Q the nuclear ones. Describe the molecule by Hamiltonian H_0:

$$H_0 = t+T+U(q, Q)+\tau(Q), \tag{1}$$

where t denotes electronic kinetic energy, $t = -\sum_j \frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_j^2}$, T nuclear kinetic energy $T = -\sum_k \frac{\hbar^2}{2M_k}\frac{\partial^2}{\partial Q_k^2}$, $U(q, Q)$ denotes the potential energy of electrons in the field of nuclei and one another, and $\tau(Q)$ denotes the mutual potential energy of the nuclei alone. One describes then the system by the Born—Oppenheimer approximation forming the total wave function $\Psi(q, Q)$ as the product of electronic wave function $\psi_n(q, Q)$ depending parametrically on Q's and vibrational wave function $\chi_{nj}(Q)$. These functions satisfy the equations

$$[t+U(q, Q)]\,\psi_n(q, Q) = E_n(Q)\,\psi_n(q, Q) \tag{2}$$

and

$$[T+\tau(Q)+E_n(Q)]\,\chi_{nj}(Q) = W_{nj}\,\chi_{nj}(Q). \tag{3}$$

2. Allow now two potential energies $E_1(Q)$ and $E_2(Q)$ to cross. The crossing will occur in a point for diatomic molecule (see Figure) and along a surface or respective hypersurface for complex molecule. (We will call it a crossing surface. irrespective of dimensions). Such crossing occurs very often in reactions involving electronically excited states [1] in predissociation, preionisation and other topics.

We remark that for points which lay on crossing surface the Born-Oppenheimer approximation is not valid. Indeed, the off-diagonal terms neglected in the Born—Oppenheimer approximation [2], [3] may be expressed in the form

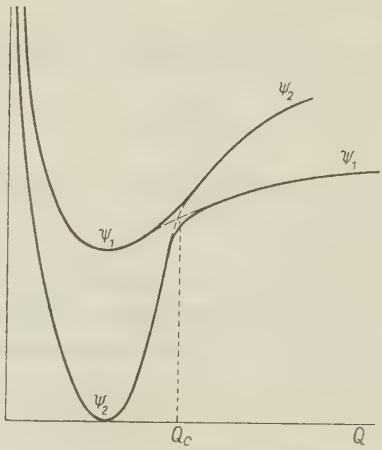
$$[E_1(Q) - E_2(Q)]^{-1}\int \psi_m^* \frac{\partial U(q, Q)}{\partial Q}\,\psi_n\,dq \tag{4}$$

and may become very large for the Q's lying on the crossing surface.

Total wave function $\Psi(q, Q)$ may be however always described in the form

$$\Psi(q, Q) = \sum_n a_n(Q)\,\psi_n(q, Q), \tag{5}$$

where ψ_n (q, Q), solutions of (3) form the complete set and a_n are vibrational amplitudes.



3. Suppose now that the crossing potential energies $E_1(Q)$, $E_2(Q)$ interact with one another. This interaction occurs owing to some neglected terms in the original Hamiltonian (1) (in the diatomic molecule such as e.g. spin-orbit coupling). Suppose the crossing potential energies $E_1(Q)$, $E_2(Q)$ to be well separated from all other potential energies $E_n(Q)$, $n > 2$, and their splitting caused by interaction to be sufficiently small to keep $E_1(Q)$, $E_2(Q)$ well separated from the other $E_n(Q)$. Under these assumptions we can reduce expansion (5) to two terms only:

$$\Psi = a_1(Q)\,\psi_1(q, Q) + a_2(Q)\,\psi_2(q, Q) \tag{6}$$

and where $a_1(Q)$, $a_2(Q)$ are to be determined. The total Hamiltonian of the system has now the form

$$H = H_0 + V(q, Q), \tag{7}$$

where $V(q, Q)$ denotes the interaction energy of $E_1(Q)$ and $E_2(Q)$. Substitute (6) to (7), multiply by the complex conjugate of ψ_1, ψ_2 and integrate over electronic coordinates; the functions ψ_1, ψ_2 as the solutions of (2) are supposed to vary slowly with nuclear coordinates, so the commutators $[\psi_1, T] = [\psi_2, T] = 0$. Supposing ψ_1, ψ_2 to be orthonormal we obtain the equations determining $a_1(Q)$, $a_2(Q)$

$$\begin{cases} [T + E_1(Q) + \tau(Q) + \langle\psi_1 | V | \psi_1\rangle]\, a_1 + \langle\psi_1 | V | \psi_2\rangle\, a_2 = \epsilon\, a_1, \\ [T + E_2(Q) + \tau(Q) + \langle\psi_2 | V | \psi_2\rangle]\, a_2 + \langle\psi_2 | V | \psi_1\rangle\, a_1 = \epsilon\, a_2, \end{cases} \tag{8}$$

where ϵ is the eigenvalue. Now, by the Hermitian property

$$\langle\psi_1 | V | \psi_2\rangle = \langle\psi_2 | V | \psi_1\rangle = g(Q) \tag{9}$$

whenever ψ_1, ψ_2 are real, as, without loss of generality, we always may suppose. Denote further

$$\begin{cases} T+E_1(Q)+\tau(Q)+\langle\psi_1|V|\psi_1\rangle = T+W_1(Q) = H_1, \\ T+E_2(Q)+\tau(Q)+\langle\psi_2|V|\psi_2\rangle = T+W_2(Q) = H_2, \end{cases} \tag{10}$$

$$a_1(Q) = \alpha(Q), \quad a_2(Q) = \beta(Q)$$

then (8) may be written in the form

$$\boldsymbol{H}\boldsymbol{\Phi} = \epsilon\boldsymbol{\Phi}, \tag{11}$$

the $\boldsymbol{H}$ has the matrix form:

$$\boldsymbol{H} = \frac{1}{2}(H_1+H_2)\mathbf{1} + \frac{1}{2}(H_1 - H_2)\boldsymbol{\sigma}_3 + g\boldsymbol{\sigma}_1, \tag{12}$$

where $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_1$ denote familiar Pauli matrices, and where $\boldsymbol{\Phi}$ is written as the column vector

$$\boldsymbol{\Phi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The eigenfunctions of (11) determine the probability amplitudes of transitions from one curve to another, determine therefore the probability of non-adiabatic reaction.

4. Eq. (12) is fundamental for the theory of non-adiabatic reactions, and from it follow, as a limiting case, the Landau—Zener [4], [5] equations. Indeed, make the following assumptions:

1) Make the WKB approximation to (11), (12); then we can write for α, β:

$$\begin{cases} \alpha = \exp\left[\frac{i}{\hbar}\int S_0\,dQ + \int S_1\,dQ\right] = e^{\frac{i}{\hbar}\int P\,dQ}\,\alpha_0, \\ \beta = \exp\left[\frac{i}{\hbar}\int \mathcal{S}_0\,dQ + \int \mathcal{S}_1\,dQ\right] = e^{\frac{i}{\hbar}\int \mathcal{P}\,dQ}\,\beta_0, \end{cases} \tag{13}$$

where $P(Q)$ and $\mathcal{P}(Q)$ denote momentum.

2) Suppose the crossing curves to interact in such a small vicinity of crossing point Q_c as to allow:

a) to retain only linear (with respect to nuclear coordinates) term in the development of $W_1(Q)$, $W_2(Q)$,

b) to consider the momentum as constant, equal its value at the crossing point Q_c.

Under these assumptions substitution of (13) to (11), (12) gives, to the first order in $\hbar$, the equations determining α_0, β_0

$$\begin{cases} i\hbar v\frac{d\alpha_0}{dQ} = l_1(Q - Q_c)\,\alpha_0 + g\beta_0, \\ i\hbar v\frac{d\beta_0}{dQ} = l_2(Q - Q_c)\,\beta_0 + g\alpha_0, \end{cases} \tag{14}$$

where v denotes classical velocity (at the crossing point), and l_1, l_2 are the slopes of W_1, W_2 at the crossing point. These are Landau—Zeuer equations.

The first assumption is valid, when curves cross at small angles and when strong interaction smoothes the angles, because WKB approximation requires slow changes of the potential. On the other hand, the second assumption is valid for curves crossing at big angles and interacting only slightly, because only then the crossing point is determined. One cannot suspect both assumptions to be satisfied simultaneously in many cases. Indeed recently [6] the Landau—Zener formula has been subject to serious criticism.

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EXPERIMENTAL PHYSICS

Spectral Investigation of Rhodamine Polymerisation Processes in Collodion

by

R. DRABENT and A. JACHYM

Presented by A. JABŁOŃSKI on June 6, 1961

The absorption spectra of rhodamine in liquid and rigid collodion solution at various concentrations of the dye were investigated.

The measurements were carried out by means of a Zeiss mirror monochromator and 931 A type electronic photomultiplier operating in spectrophotometric circuit and using plane-parallel vessels made at this laboratory.

Rhodamine B extra from B.D.H. Laboratory Chemicals Group and collodion in accordance with Polish Pharmacopoea III (a 4% solution of nitrocellulose in an ether-alcohol mixture) [1] were used.

For investigation at high concentrations of the dye, rigid solutions were employed. These were prepared by evaporating liquid solutions. The concentration of the rigid solutions was referred to the volume of the sample in the solid state. Since measuring the changes in volume of the samples in the process of solidification involves rather considerable error (amounting to about 24%), the concentrations of rhodamine in the rigid solutions as given in Table II and Fig. 2 are approximate. The same error results in computing the optical densities D (Fig. 2).

The absorption spectra were measured in liquid solutions throughout the concentration range of 10^{-6} to 10^{-3} g/ml and in the rigid solutions from 2.5×10^{-3} to 5×10^{-2} g/ml.

The experimental results for the liquid solutions are assembled in Fig. 1 and Table I. Fig. 1 gives the dependence of the optical density D on the wavelength for the extreme values of the concentration. The other curves are distributed regularly between the foregoing two.

On comparing the absorption spectra with those of rhodamine B extra in water and alcohol (Table III), the long wave maximum (at $\lambda = 5440 \div 5400$ Å) is seen to be shifted towards shorter wavelengths with respect to the absorption maximum of rhodamine monomers in aqueous and alcohol solutions. As the concentration increases, the height of the long wave maximum diminishes slightly and that of the short wave maximum increases (Fig. 1).

These data lead us to ascribe the long wave absorption maximum to rhodamine monomers in collodion. The experimental shift towards shorter wavelengths with respect to the absorption maximum in aqueous solution can be explained by the effect of the solvent.

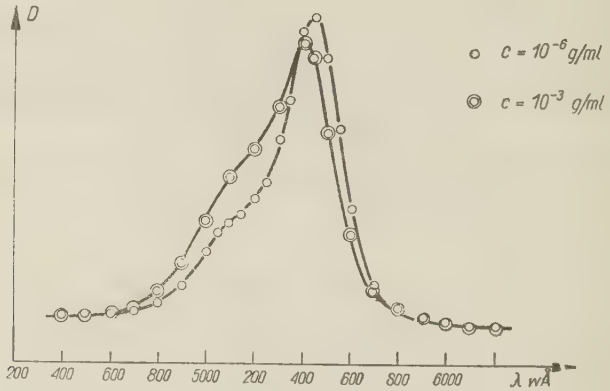

Fig. 1.

On the other hand, it would seem that the short wave maximum increasing towards higher concentrations is due to the presence of rhodamine dimers. The latter maximum, too, is displaced towards shorter wavelengths with respect to the absorption maximum of rhodamine dimers in water (Table III).

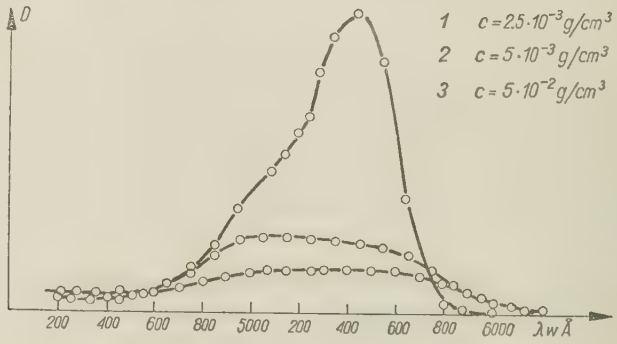

Fig. 2.

The concentrational effects can also be related to ionisation processes in the dye molecule, also involving changes in the absorption spectra in the visible region. In the case of rhodamine, however, no such process occurs [4]. The concentrational variations in the absorption spectra of rhodamine would seem to result from dimerisation processes [2]—[4].

The absorption spectra of rhodamine in rigid solution are shown in Fig. 2. The wavelengths corresponding to the absorption maxima at different concentrations of the dye are assembled in Table II. In cases when the absorption maximum was not sharply outlined, the values in Table II can serve but for orientation.

TABLE I

(Liquid solutions)

Concentration in g/cm.^{3}	10^{-6}	1.6×10^{-6}	6×10^{-6}	10^{-5}	5×10^{-5}	10^{-4}	5×10^{-4}	10^{-3}
Wavelength in Å, of long wave absorption maximum	5440	5440	5440	5440	5450	5430	5400	5400
Wavelength, in Å, of short wave absorption maximum	about 5100							

TABLE II

(Rigid solutions)

Concentration in g/cm.^{3}		5×10^{-2}	5×10^{-3}	2.5×10^{-3}
Wavelength, in Å, of absorption maximum	long wavelength maximum	5600	5560	5450
	short wavelength maximum	5000	5000	5000

TABLE III

Rhodamine 3B extra

Solvent		Collodion liquid solution *)	Water **)	Alcohol or water ***)
Wavelength, in Å, of absorption maximum	monomers	5440	5540	5500
	dimers	5100	5240, 5590	

For the solution of dye concentration 2.5×10^{-3} g/cm^3, the spectrum is essentially of the same type as in liquid solution (Fig. 2, curve 1 and Fig. 1); this is true of the shape of the spectrum, the position of the absorption maximum of the monomers, and the value of the absorption at absorption maximum (the differences are insignificant).

*) Present paper, **) according to [2], ***) according to [5].

However, already at the twice greater concentration of 5×10^{-3} g/cm^3, the absorption spectra change visibly (Fig. 2, curves 2 and 3). Quite generally, absorption becomes lesser and two equivalent maxima appear.

The short wave maximum (about $\lambda = 5000$ Å) seems to be due to rhodamine molecule dimers. Two facts point to this conclusion:

1. the maximum under consideration appears in solutions of high concentration of the dye,
2. it corresponds to the short wave absorption maximum in liquid solutions which, too, we also ascribed to dimers.

The short wave maximum ($\lambda = 5000$ Å) of the rigid solution of concentration 2.5×10^{-3} g/cm^3 (Fig. 2, curve 1) should also be included in this cathegory and ascribed to dimers of the molecule of the dye.

The insignificant shift in the absorption maximum which we ascribed to dimers in the liquid and rigid solutions would seem to be fully explained by the change in the medium due to the changes occurring in the composition of collodion in the process of evaporation.

The long wave maximum ($\lambda = 5550$—5600 Å) appearing at high concentrations of the rhodamine (Fig. 2, curves 2 and 3) is markedly displaced towards longer wavelengths in comparison with that of the monomers (by about 130 Å). At such high concentrations of the dye ($5\times10^{-3} \div 5\times10^{-2}$ g/cm^3), monomers should account for only a small fraction of all rhodamine molecules, in accordance with the law of masses. Thus, their participation in the absorption is small and the absorption spectrum can fail to reveal a clear maximum corresponding to them. This would justify explaining the long wave maximum (Fig. 2, curves 2 and 3) as also being due to rhodamine dimers. Such a long wave absorption maximum for dimers, albeit much weaker than the short wave maximum, was found by Forster for rhodamine in aqueous solution [2].

From the absorption spectra of Fig. 2 (curves 2 and 3) it is not easy to determine precisely the position of the two absorption maxima ascribed to rhodamine dimers in collodion, because of the flattened shape of the absorption curves and possible superposition of the weak monomer band on the absorption spectrum of the dimers.

Conclusions

1. Rhodamine in collodion solution was found to form dimers.

2. In liquid solutions, processes of dimerisation of rhodamine molecules occur but to a very small degree, as shown by the small variations in the absorption throughout the concentration range of 10^{-6} to 10^{-3} g/cm.^{3}.

3. The absorption maximum of rhodamine dimers in liquid solutions occurs at $\lambda = 5100$ Å, i.e. to the short wave side of the monomer band.

4. The absorption spectrum of rhodamine in rigid solution of concentration 2.5×10^{-3} g/cm^3 is essentially the same as in liquid solution.

5. At 5×10^{-3} g/cm^3 and higher concentrations, very great changes occur in the absorption spectrum of rhodamine in rigid solutions. A broad, flat band appears revealing two absorption maxima, which would seem to be due to rhodamine dimers.

6. The absorption maximum of rhodamine monomers in collodium corresponds to a wavelength of 5440 Å both for the liquid and rigid solutions.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

ТОМ IX Резюме статей ВЫПУСК 9

С. ГЛИДМАН, **О НЕКОТОРЫХ ГЕОМЕТРИЧЕСКИХ СВОЙСТВАХ НОРМАЛЬНЫХ МАТРИЦ** . . . . . стр. 625—631

1. Пусть A обозначает нормальную вещественную матрицу, имеющую характеристические собственные значения $\lambda_1, \lambda_2, \ldots, \lambda_n$, упорядоченные таким образом, что $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$,

Доказано, что

$$|\lambda_1, \lambda_2, \ldots, \lambda_p| = \lim_{k\to\infty} \sqrt[2k]{A^{2k}_{p,a_p}}, \tag{1}$$

где A^{2k}_{p,a_p} обозначает главный минор порядка p матрицы A^{2k} не меньший, чем остальные главные миноры порядка p этой матрицы.

Можно доказать, что последовательность в формуле (1) — неубывающая.

2. Доказано, что для достаточно больших значений k порядок следования столбцов матрицы A^k, согласно их длине, имеет устойчивый характер, не зависящий от k, и введено понятие дефинитивно максимального столбца матрицы A.

Доказано, что дефинитивно максимальный столбец симметрической матрицы не ортогонален, по крайней мере, к одному собственному вектору, соответствующему собственному значению, наибольшему по абсолютной величине.

3. Доказано, что очередное p-тое собственное значение нормальной матрицы может быть выражено формулой

$$|\lambda_p| = \lim_{k\to\infty} \sqrt[k]{h_{kp}(\gamma)}, \tag{2}$$

где $h_{kp}(\gamma)$ — длина последовательной p-той высоты матрицы A^k (сравн. [3], [5]), определенной при условии, что столбцы этой матрицы рассматриваются в некотором порядке, называемом регулярным порядком.

Показано, каким путем можно эффективно найти регулярный порядок неособенной симметрической матрицы, имеющей собственные значения, различные по абсолютной величине.

4. Доказано, что при неограниченно возрастающем k, направления последовательных высот $h_{kp}(\sigma)$ матрицы S^k [5], где S — неособенная симметрическая матрица, имеющая собственные значения, разные по абсолютной величине — параллельны к очередным собственным векторам $\bar{x}_p$ этой матрицы.

5. Доказано, что последовательность матриц, полученная путем применения алгорифма LR Рутисгаузера [9], к определенному однозначному разложению регулярной неособенной симметрической матрицы, имеющей собственные значения, разные по абсолютной величине, на произведение матриц: ортогональной и трехугольной верхней — сходится и ее пределом является диагональная матрица, подобная данной матрице.

М. АЛЬТМАН, **О МЕТОДЕ КАСАТЕЛЬНЫХ ГИПЕРБОЛ ДЛЯ ОПРЕАТОРНЫХ УРАВНЕНИЙ** . . . . . стр. 633—637

В настоящей заметке исследуется метод касательных гипербол для решения операторных уравнений. Для этой цели используется мажорантный принцип, разработанный Л. В. Канторовичем. Такая же идея была применена автором для исследования другого варианта метода касательных гипербол.

М. АЛЬТМАН, **ИТЕРАЦИОННЫЙ МЕТОД ДЛЯ СОБСТВЕННЫХ ЗНАЧЕНИЙ И СОБСТВЕННЫХ ВЕКТОРОВ МАТРИЦ** . . . . . стр. 639—644

Работа содержит некоторый итерационный метод нахождения приближенных собственных значений и собственных векторов симметричных матриц. Этот метод является частным случаем предложенного автором общего метода решения нелинейных функциональных уравнений.

Сущность последнего метода состоит в обобщении классического метода Ньютона для нахождения нулевых элементов нелинейных функционалов. Таким образом, мы имеем здесь дело с методом типа Ньютона. Замечательной чертой этого метода является то, что он всегда сходится, в отличие от классического метода Ньютона, который этим свойством не обладает.

М. АЛЬТМАН, **ОБЩИЙ МЕТОД НАИСКОРЕЙШЕГО ОРТОГОНАЛЬНОГО СПУСКА** . . . . . стр. 645—651

В работе содержится итерационный метод решения линейных уравнений в пространстве Гильберта. Характерной чертой этого метода является то, что поправки, улучшающие приближенное решение, ортогональны к свободному вектору уравнения и к некоторому подпространству, которое может быть выбрано многими способами. От свободного вектора и от выбора подпространства зависит скорость сходимости метода. При удачном выборе скорость может быть гораздо больше, чем в методе наискорейшего спуска.

Кроме того, дается применение метода к решению линейных алгебраических уравнений.

М. АЛЬТМАН, **МЕТОД РЕЛАКСАЦИИ ГРАДИЕНТА ДЛЯ ЛИНЕЙНЫХ АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ** . . . . . стр. 653—657

Вместо метода наискорейшего спуска вдоль градиента в работе предлагается метод наискорейшего спуска вдоль другого направления, полученного путем релаксации по координатам градиента с таким расчетом, чтобы получить — по мере возможности — большее уменьшение значения функционала меры ошибки. Таким образом, итерационный процесс состоит из основных итераций спуска, для каждой из которых применяется один или несколько шагов релаксации по координатам градиента.

Р. ТАБЕРСКИЙ, **НЕКОТОРЫЕ СВОЙСТВА (K, φ)-СУММИРОВАНИЯ РЯДОВ** стр. 659—666

В работе приводится несколько результатов, касающихся нового метода суммирования рядов, предложенного П. П. Коровкиным [1]. Выяснена связь между методами Чезаро и Коровкина, а также указан порядок сходимости множителей и соответственных полиномиальных операторов.

А. ЛЯСОТА и З. ОПЯЛЬ, **ОБ ИНТЕРПОЛЯЦИОННОЙ ПРОБЛЕМЕ ДЛЯ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ n-ТОГО ПОРЯДКА** . стр. 667—671

Авторы рассматривают дифференциальное уравнение n-того порядка

$$x^{(n)} = f(t, x, x', ..., x_{n-1}) \tag{1}$$

и при предположении, что непрерывная функция $f(t, x_0, x_1, ..., x_{n-1})$ выполняет в множестве: $a < t < b$, $-\infty < x_i < +\infty$ $(i = 0, 1, ..., n-1)$ неравенство

$$|f(t, x_0, x_1, ..., x_{n-1})| \leqslant M + \sum_{i=0}^{n-1} L_i |x_i| \qquad (M \geqslant 0, L_i > 0)$$

доказывают следующую теорему:

Если для произвольной последовательности $t_1, ..., t_n$ чисел, принадлежащих к интервалу (a, b) функция $x(t) = 0$ является единой функцией, выполняющей дифференциальное неравенство

$$|x^{(n)}(t)| \leqslant \sum_{i=0}^{n-1} L_i |x^{(i)}(t)| \qquad (a < t < b)$$

и условия $x(t_1) = ... = x(t_n) = 0$, то для произвольных n-точек $(t_1, c_1), ..., (t_n, c_n)$ $(a < t_1 < t_2 < ... < t_n < b)$ имеется по крайней мере одно решение $x(t)$ уравнения (1), удовлетворяющее условиям $x(t_1) = c_1, ..., x(t_n) = c_n$.

З. ОПЯЛЬ, **ДИФФЕРЕНЦИАЛЬНОЕ ПОЧТИ ПЕРИОДИЧЕСКОЕ УРАВНЕНИЕ, НЕ ИМЕЮЩЕЕ ПОЧТИ ПЕРИОДИЧЕСКИХ РЕШЕНИЙ** . . . . . стр. 673—676

Автор дает пример дифференциального уравнения первого порядка

$$w' = B(\varphi, w) \tag{1}$$

с правой почти периодической стороной по аргументу φ, все решения которого ограничены в пределе $(-\infty, +\infty)$. Ни одно из этих решений не является, однако, почти периодической функцией. В конструкции уравнения (1) автор использует свойства интегралов дифференциальных уравнений, определенных на торе.

Ч. БЕССАГА, А. ПЕЛЧИНСКИЙ и С. РОЛЕВИЧ, **О ПОПЕРЕЧНИКОВОЙ АППРОКСИМАТИВНОЙ РАЗМЕРНОСТИ И ЛИНЕЙНОЙ ОДНОРОДНОСТИ *F*-ПРОСТРАНСТВ** стр. 677—683

1) Исследуется понятие поперечниковой аппроксимативной размерности линейного топологического пространства. При помощи этого понятия приводится пример бесконечномерного *F*-пространства не изоморфного своему подпространству дефекта 1.

2) Доказывается следующая теорема. *Всякое бесконечномерное, ненормируемое F-пространство, которое неизоморфно произведению пространства Банаха на пространство s, содержит бесконечномерное, ядерное подпространство, изоморфное пространству* $k \ni m \ni$.

К. БОРСУК, **ЗАМЕЧАНИЯ О РАЗМЕРНОСТИ МНОЖЕСТВ** ANR . . .стр. 685—687

Предметом работы является доказательство следующей теоремы:

Теорема. *Пусть* $\{K_\alpha\}$ *— семейство n-размерных множеств* ANR, *лежащих в некотором пространстве X, являющемся тоже множеством* ANR. *Если индекс* α *проходит несчетное множество A, то существуют два разные индексы* $\alpha, \alpha' \in A$ *такие, что* $\dim (K_\alpha \cap K_{\alpha'}) = n$.

Очевидно, что теорема не будет справедливой, если пренебречь предположением, что пространство X является множеством ANR. Однако, остается открытой проблема — можно ли пренебречь предположениями, что множества K_α являются множествами ANR.

С. ГОРГОЛЕВСКИЙ, Я. ХАНАШ, Г. ИВАНИШЕВСКИЙ, З. ТУРЛО, **ПРИМЕНЕНИЕ ЛОГ-ПЕРИОДИЧЕСКИХ АНТЕНН К РАДИОАСТРОНОМИИ** . стр. 689—691

Были применены лог-периодические антенны к солнечному радиоинтерферометру. Две такие антенны, построенные для работы в диапазоне 100-1000 *Мгц*, подвергались испытаниям. На частотах 127 *Мгц* и 327 *Мгц* прием излучения спокойного Солнца оказался удовлетворительным.

Построенный интерферометр дает возможность принимать на неподвижных антеннах радиоизлучение Солнца в течение семи часов в сутки.

М. ДЕМЯНСКИЙ и Э. ИНФЕЛЬД, **ЗАМЕТКА О МЕТОДЕ ПОЛЯ ДЛЯ ПОЛУЧЕНИЯ ЗАКОНОВ СОХРАНЕНИЯ И ДЛЯ РЕШЕНИЯ ЗАДАЧИ ДВУХ ТЕЛ В ТЕОРИИ ОБЩЕЙ ОТНОСИТЕЛЬНОСТИ** . . . . . стр. 693—696

В работе законы сохранения вплоть до пост-Ньютоновского приближения выведены авторами непосредственно из уравнений поля. Таким образом, получен простой метод решения уравнений движения для задачи двух тел, не прибегая к использованию Лагранжиана и не записывая его в открытом виде („explicite").

А. ВИТКОВСКИЙ, **О ПЕРЕХОДАХ С ПЕРЕСЕЧЕНИЕМ ПОВЕРХНОСТЕЙ ПОТЕНЦИАЛЬНОЙ ЭНЕРГИИ** . . . . . стр. 697—700

Показано, что в точках пересечения поверхностей потенциальной энергии для движений ядер в молекулах приближение Борна-Оппенгеймера не является справедливым. Выведены основные уравнения для амплитуд вероятности перехода между пересекающимися поверхностями потенциальной энергии. Показано, что — как весьма особый случай — вытекает на основании настоящей теории формула Ландау-Ценера.

Р. ДРАБЕНТ и А. ЯХЫМ, **СПЕКТРАЛЬНЫЕ ИССЛЕДОВАНИЯ ПРОЦЕССОВ ПОЛИМЕРИЗАЦИИ РОДАМИНА В КОЛЛОДИИ** . . . . . стр. 701—705

Исследованы спектры поглощения родамина в жидком и твердом растворах коллодия с различной концентрацией красителя.

Констатировано, что родамин в этих растворах образует димеры. Процесс димеризации частиц родамина выступает отчетливо лишь при концентрации $5 \cdot 10^{-3} г/см^3$ и большей. Димеры частиц родамина обладают широкой полосой поглощения с двумя максимумами, которые лежат по обеим сторонам максимума поглощения мономеров.

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